

Price of Competition and Dueling Games

Sina Dehghani ^{*†}

MohammadTaghi HajiAghayi ^{*†}

Hamid Mahini ^{*†}

Saeed Seddighin ^{*†}

Abstract

We study competition in a general framework introduced by Immorlica, Kalai, Lucier, Moitra, Postlewaite, and Tennenholtz [19] and answer their main open question. Immorlica et al. [19] considered classic optimization problems in terms of competition and introduced a general class of games called *dueling games*. They model this competition as a zero-sum game, where two players are competing for a user’s satisfaction. In their main and most natural game, the *ranking duel*, a user requests a webpage by submitting a query and players output an ordering over all possible webpages based on the submitted query. The user tends to choose the ordering which displays her requested webpage in a higher rank. The goal of both players is to maximize the probability that her ordering beats that of her opponent and gets the user’s attention. Immorlica et al. [19] show this game directs both players to provide suboptimal search results. However, they leave the following as their main open question: “does competition between algorithms improve or degrade expected performance?” (see the introduction for more quotes) In this paper, we resolve this question for the ranking duel and a more general class of dueling games.

More precisely, we study the quality of orderings in a competition between two players. This game is a zero-sum game, and thus any Nash equilibrium of the game can be described by minimax strategies. Let the value of the user for an ordering be a function of the position of her requested item in the corresponding ordering, and the social welfare for an ordering be the expected value of the corresponding ordering for the user. We propose the *price of competition* which is the ratio of the social welfare for the worst minimax strategy to the social welfare obtained by a social planner. Finding the price of competition is another approach to obtain structural results of Nash equilibria. We use this criterion for analyzing the quality of orderings in the ranking duel. Although Immorlica et al. [19] show that the competition leads to suboptimal strategies, we prove the quality of minimax results is surprisingly close to that of the optimum solution. In particular, via a novel factor-revealing LP for computing price of anarchy, we prove if the value of the user for an ordering is a linear function of its position, then the price of competition is at least 0.612 and bounded above by 0.833. Moreover we consider the cost minimization version of the problem. We prove, the social cost of the worst minimax strategy is at most 3 times the optimal social cost.

Last but not least, we go beyond linear valuation functions and capture the main challenge for bounding the price of competition for any arbitrary valuation function. We present a principle which states that the lower bound for the price of competition for all 0-1 valuation functions is the same as the lower bound for the price of competition for all possible valuation functions. It is worth mentioning that this principle not only works for the ranking duel but also for all dueling games. This principle says, in any dueling game, the most challenging part of bounding the price of competition is finding a lower bound for 0-1 valuation functions. We leverage this principle

^{*}University of Maryland. email: {seddighin,dehghani,mahini}@umd.edu, hajiagha@cs.umd.edu

[†]Supported in part by NSF CAREER award CCF-1053605, NSF BIGDATA grant IIS-1546108, NSF AF:Medium grant CCF-1161365, DARPA GRAPHS/AFOSR grant FA9550-12-1-0423, another DARPA SIMPLEX grant, and a Google Faculty Research award.

to show that the price of competition is at least 0.25 for the generalized ranking duel, and to find upper bounds on the price of competition for the binary search duel and the compression duel which are introduced by Immorlica et al. [19].

1 Introduction

The conventional wisdom is that competition among suppliers will increase social welfare by providing consumers with competitive prices, high-quality products, and a wide range of options. A classic example is the Bertrand competition [10] where suppliers compete in price to incentivize consumers to buy from them and as a result the market price decreases to the point that it matches the marginal cost of production. Indeed there are many theoretical and empirical studies for supporting this belief in the economic literature (See, e.g., [2, 29, 25, 24]). However while in many markets the competition steers businesses to optimize their solutions for consumers, there are competitive markets in which businesses do not offer the best option to consumers. An interesting example for describing this situation is a dueling game, namely, a zero-sum game where two players compete to attract users. Immorlica, Kalai, Lucier, Moitra, Postlewaite, and Tennenholtz [19] showed surprisingly if players are aimed to beat their opponents in a dueling game, they may offer users suboptimal results. However, they raised this question regarding the efficiency of the competition as the authors write, “*Perhaps more importantly, one could ask about performance loss inherent when players choose their algorithms competitively instead of using the (single-player) optimal algorithm. In other words, what is the price of anarchy¹ of a given duel? ... Our main open question is (open question 1): does competition between algorithms improve or degrade expected performance?*” As we describe below, we study this open question for a set of dueling games and in particular for the ranking duel which is an appropriate representative of dueling games due to Immorlica et al. [19].

Dueling games. A *dueling game* \mathcal{G} is a zero-sum game where two players compete for the attention of a user². In a dueling game both players try to beat the other player and offer a better option with a higher value to the user. In particular, while the user’s request is unknown to both players and they only have access to probability distribution p , the goal for each player is to maximize the probability that her offer is better than her opponent’s offer. This framework falls within a general and natural class of ranking or social context games [7, 11], where each player plays a base game separately and then ultimate payoffs are determined by both their own outcomes and the outcomes of others. Immorlica et al. argue that this class of games models a variety of scenarios of competitions between algorithm designers, such as, competition between search engines (who must choose how to rank search results), or competition between hiring managers (who must choose from a pool of candidates in the style of the secretary problem).

To be more precise a dueling game is defined by 4-tuple $\mathcal{G} = (\Omega, p, S, v)$, where Ω is the set of all possible requests from the user, p is a probability distribution over set Ω i.e., p_ω is the probability of requesting $\omega \in \Omega$ by the user, S is the set of all possible pure strategies for both players, and $v_\omega(s)$ is the value of pure strategy $s \in S$ for the user upon request $\omega \in \Omega$. Note that v is usually considered to be the valuation of the players, but in this paper valuation function v denotes the value for the user. While a mixed strategy is a probability distribution over all possible pure strategies in S , we write the value of mixed strategy \mathbf{x} as $v_\omega(\mathbf{x}) = E_{s \sim \mathbf{x}}[v_\omega(s)]$.

A social planner is often interested in choosing a strategy which maximizes the social welfare, even though it may be a bad strategy in the competition between players. This means the social

¹Indeed Immorlica et al. [19] use the term of the *price of anarchy* in their aforementioned open question for the same concept of the *price of competition* in this paper.

²One can see the user as a population of users with the same behavior.

welfare maximizer strategy may not appear in any Nash equilibrium of the game, and thus the competition between players results in a suboptimal outcome for the users. Knowing the fact that Nash equilibria of a dueling game can be formed by suboptimal strategies, the following question seems to be an important question to ask regarding the inefficiency of this competition:

What is the social welfare of any Nash equilibrium in a dueling game in comparison to the social welfare of the optimal strategy?

Price of competition. As aforementioned while in so many cases the competition motivates businesses to optimize their solutions for consumers, there are competitive markets and in particular dueling games of Immorlica et al. [19], in which businesses do not offer the best option to consumers. We define *price of competition* in this paper to capture this phenomenon.

First we note that since dueling games are two-player zero-sum games, Nash equilibria of these games are characterized by minimax strategies. Therefore, one can measure the inefficiency of any Nash equilibrium by comparing the welfare of any minimax strategy, in a game of competition between two players, with the welfare achieved by a social welfare maximizer. We are now ready to define the following criterion for measuring the quality of minimax strategies in a dueling game.

Definition 1.1 Price of competition (PoC) *is the ratio between the social welfare of the worst minimax strategy and the social welfare of the best possible strategy.*

The proposed concept of the price of competition has the same spirit as the concept of the price of anarchy, and both concepts try to measure the inefficiency of Nash equilibria quantitatively. The price of anarchy, introduced by the seminal work of Koutsoupias and Papadimitriou [23], is a well-known concept in game theory that measures the ratio of the social welfare of the worst Nash equilibrium to the optimal social welfare. Although these two concepts are defined to capture properties of Nash equilibria, they are meaningfully different. In the price of anarchy, the social welfare is defined as the expected utility of all players in an equilibrium “outcome”³ which is always zero for any zero-sum game. However, in the price of competition, the social welfare is the expected utility of the user (which is not a player) in a minimax “strategy”. In fact, the price of competition is aimed to analyze the impact of the competition between players on an external user.

Since the price of competition captures the inefficiency of minimax strategies in two-player zero-sum games and all Nash equilibria of any two-player zero-sum game can be described by the set of minimax strategies, we believe the price of competition sheds new light on the structural analysis of Nash equilibria in two-player zero-sum games. Indeed as Alon, Demaine, Hajiaghayi, and Leighton [5] mention understanding the structure of Nash equilibria, and not just the price of anarchy, is very important in general and thus our work is exactly toward this direction.

Due to the space constraints, all the missing proofs are provided in the appendices.

1.1 Our results

Ranking duel: To define the ranking duel more precisely, consider a ranking duel with two players. When a user submits a query to a player, she is basically searching a webpage which is unknown to the player. The player only has a prior knowledge about the requested webpage, i.e., for each webpage the probability that this webpage is requested by the user is known. The strategy of each player is an ordering for displaying webpages. When the requested webpage is realized, the player which puts this webpage in a higher rank gets the user attention, and thus wins the competition.

³Which is essentially the same as the sum of utilities of all players.

The goal of each player is to maximize the probability of winning the competition. In this situation, a social planner who wants to minimize the expected rank of the requested webpage lists webpages in a decreasing order of their probabilities. However, this strategy may lose the competition to another strategy.⁴

We first investigate the quality of minimax strategies and prove that surprisingly the social welfare of any minimax strategy is not far from that of the optimal solution; it is 0.612 of the optimal solution for the linear valuation functions and 0.25 of the optimal for any arbitrary valuation function.

Theorem 1 *Consider an instance of the raking duel. If the valuation function is a non-negative linear function of the rank, the price of competition is at least 0.612 for $|\Omega| \geq 10$, and at most 0.833.*

Our proof needs a careful understanding of properties for minimax strategies and has three main steps. First, we prove nice structural properties of minimax strategies. This step is the main step toward bounding the price of competition and gives an insight into properties of the polytope of minimax strategies. For example for every two webpages ω_1 and ω_2 with $p_{\omega_1} > p_{\omega_2}$, we prove there is a lower bound on the probability that any minimax strategy ranks webpage ω_1 before webpage ω_2 . In the next step, we leverage these properties to write a factor-revealing mathematical program for bounding PoC. At last, we find a linear program where the set of its feasible solutions is a superset of the set of feasible solutions of the former mathematical program. We find the optimal solution of this linear program to formally prove the theorem for $|\Omega| \geq 10$. Moreover, we write a computer program to find the optimal solution of the corresponding linear program and show the price of competition is at least 0.637 for $|\Omega| \geq 100$ (which is slightly better the case that $\Omega \geq 10$). To the best of our knowledge, we are the first to use factor-revealing techniques to bound the inefficiency of equilibria.

Afterwards, we consider the cost minimization version of the ranking duel and prove a constant upper bound for the social cost of the game using the same technique of Theorem 1. Note that the only difference between the cost minimization and welfare maximization of a dueling game is that function v is a cost function rather than a valuation function, and once a webpage is searched the winner of the cost minimization game is the player who provides a solution with a lower cost. Moreover, we define the PoC_{cost} of the ranking duel as the ratio between the minimax strategy with the highest cost and the strategy with the least cost. In the following theorem we show that $PoC_{cost} \leq 3$.

Theorem 2 *For a ranking duel with a linear cost function, we have $PoC_{cost} \leq 3$.*

It is worth mentioning that the structural properties of minimax strategies do not depend on the valuation function, and thus the polytope of minimax strategies remains unchanged for every valuation function which is a decreasing (increasing) function of rank in the welfare maximization (cost minimization) variant of the game. Therefore, we leverage structural properties of the polytope of minimax strategies, which is presented in Theorem 1, for proving Theorem 2 and in general one can apply our techniques for characterizing the polytope of minimax strategies for an arbitrary valuation function. Nevertheless, writing the factor-revealing mathematical program totally depends on the linearity of the valuation function.

⁴For example consider a situation when the user submits a query and she is interested in webpages w_1 , w_2 , and w_3 with probabilities 0.35, 0.33 and 0.32 respectively. In this situation the social planner ranks webpage w_i at position i , for $i = 1, 2, 3$. However, if a player plays based on this strategy, her opponent puts webpages w_2 , w_3 , and w_1 at positions 1, 2, and 3 respectively, and thus wins the competition when the user requests webpages w_2 or w_3 . This means the social planner strategy loses the competition with probability 0.65.

General valuation functions: There are situations where the value of the user is not a linear function of rank. For example, consider a user that only cares about the top search results and will be satisfied if and only if her requested webpage is ranked higher than a certain threshold. We investigate the efficiency of minimax strategies for any non-negative non-linear valuation function. Moreover, we go beyond the ranking duel and consider other dueling games, in the pioneering work of [19]. While bounding the social welfare for arbitrary valuation functions and general dueling games seems to be challenging, we present a general principle to capture the main challenge of this problem. The proposed principle has the same spirit as the classic 0-1 principle in the sorting network which states: “a sorting network will sort any given input if and only if it sorts any given 0-1 input [13].” The following principle has the same message and shows if one can bound the social welfare for any 0-1 valuation function, the same bound holds for any arbitrary valuation function. This means the main challenge for bounding the social welfare is to bound it for 0-1 valuation functions. The main idea for proving Theorem 3 is to decompose any valuation function into 0-1 valuation functions.

Theorem 3 *0-1 Principle: Consider a dueling game. If the price of competition is greater than α when the social welfare is defined based on any 0-1 valuation function, then it is greater than α when the social welfare is defined based on any valuation function.*

One can leverage this principle to analyze the efficiency of competition in any dueling game. For example, we show that the price of competition in the ranking duel is at least 0.25 for an arbitrary valuation function.

Theorem 4 *The price of competition is at least 0.25 for the ranking duel, when the social welfare is defined based on an arbitrary valuation function.*

In the proof of Theorem 4, based on the 0-1 principle, we first consider the problem with pseudo-valuation functions in which the value of each position is either 0 or 1. We consider \mathbf{x}^* as the minimax strategy with the least social welfare and construct a response strategy \mathbf{x}'_i for the second player for every $1 \leq i \leq n$ as follows:

Draw a permutation randomly based on strategy \mathbf{x}^* . If the value of the position of the i -th webpage is 1 then play that permutation. Otherwise, swap the position of the i -th webpage with one of the positions with value 1 at random and play the new permutation.

Next, we use the fact that minimax strategy \mathbf{x}^* does not lose to strategy \mathbf{x}'_i for proving a set of inequalities which later on helps us to bound the price of competition. Finally, we use the 0-1 principle to show that this lower bound holds for all possible valuation functions.

This principle also helps us to provide upper bounds on the price of competition when one considers a general valuation function. For example we show that the PoC of the following two games introduced by Immorlica et al. [19] cannot be bounded by any constant value:

- *Binary search duel:* The binary search duel is a dueling game where each player chooses a binary search tree over the set of all possible requests Ω . When the user’s request $\omega \in \Omega$ is realized, the value for each strategy is defined based on the depth of request ω in the corresponding binary search tree.
- *Compression duel:* The compression duel is a dueling game where each player chooses a binary tree over the set of all possible requests Ω , i.e., the set of all leaves of the binary tree would be equal to the set of all possible requests. When the user’s request $\omega \in \Omega$ is realized, the value for each strategy is defined based on the depth of request ω in the corresponding binary tree.

Theorem 5 *The price of competition is $\mathcal{O}(\frac{1}{|\Omega|})$ for the binary search duel and unbounded for the compression duel, when the social welfare is defined based on an arbitrary valuation function.*

In order to construct bad instances for these duels, we design a valuation function which is 1 for low depths, 0 for high depths, and a small positive value ϵ in between. We show the price of competition is less than any given number $\beta > 0$ for the binary search duel by constructing an instance of the binary search duel with $|\Omega| = \Theta(\frac{1}{\beta})$. However, we present an instance of the compression duel with a constant size set Ω and a price of competition less than any given value $\beta > 0$.

1.2 Related work

Immorlica et al. [19] are the first who considered the concept of dueling games. They present dueling games in the context of dueling algorithms, where two competitive algorithms try to maximize the probability of outperforming their opponent for an unknown stochastic input. While we employ the same model in this paper, our goal completely differs from that of Immorlica et al. [19]. Immorlica et al. [19] present polynomial-time algorithm for finding a minimax strategy of a dueling game when the polytope of minimax strategies can be represented by a polynomial number of linear constraints. Knowing the fact that the polytope of minimax strategies of any ranking duel has polynomially many facets, they propose a polynomial-time algorithm for finding a minimax strategy of ranking duels. This method was later generalized by [3] to solve the Colonel Blotto game. Immorlica et al. [19] leave the problem of analyzing the social welfare of competitive algorithms as their main open question. In this paper, we do not deal with the computational complexity of finding minimax strategies, but we focus on answering the posted open question and analyze the social welfare of minimax strategies for a given duel.

As we are interested in quantifying the inefficiency of Nash equilibria, our proposed concept of the price of competition has the same flavor as the concept of the price of anarchy [23, 26]. The price of anarchy is commonly used for quantifying the inefficiency of a system which is constructed by selfish agents. For example, it has been used to analyze the inefficiency of Nash equilibria in congestion games [27, 12], network creation games [16, 14, 4, 5], and selfish scheduling games [6, 20]. (See, e.g. [26] for more examples).

Kempe and Lucier [21] recently study the impact of competition on the social welfare in a competitive sponsored search market. In their model, which is a departure from the model of Immorlica et al. [19], search engines again compete to obtain more users. A user's request is defined by a set S of webpages which is unknown to search engines, and the user is satisfied if and only if at least one of webpages in S is ranked in a better position than a given threshold t . The strategy of each search engine is an ordering over all possible webpages. At last, the user chooses a search engine based on a selection rule which is a function of probability of being satisfied by each search engine. Kempe and Lucier [21] prove that if search engines extract utility from satisfied users or the search engine selection rule is convex, then the social welfare of the game is at least half of the optimum social welfare. Moreover, they show if the utility of search engines is driven from all customers and the search engine selection rule is concave, then the social welfare of the game is bounded away from that of the optimum solution by a factor of $\Omega(n)$, where n is the number of all possible webpages. We would like to note that our model is a general model for studying all dueling games which is exactly the same as the model of Immorlica et al. [19], and is significantly different from that of Kempe and Lucier [21].

There is a line of research that study a competition between advertisers in sponsored search auctions [1, 9, 17, 15, 22]. These works analyze the revenue of a single search engine in various settings regarding users' behavior and the business model of advertisers. However, in ranking duel

we investigate a competition between players who provide orderings rather than advertisers.

There is a rich literature in economics that explains product differentiation in competitive markets. While producing similar products is supported by classical models such as the Hotelling model [18], Aspremont, Gabszewicz, and Thisse [8] argue that competitive producers may improve their revenue by producing different products. See, e.g., [29, 25, 24] for details on this literature. The same phenomenon can be seen in the sponsored search market, e.g., Telang, Rajan, and Mukhopadhyay [28] show low-quality search engines may extract revenue from the sponsored search market.

2 Model

2.1 Dueling games

In dueling game \mathcal{G} both players try to beat the other player and offer a better value in the competition. Assume players A and B play pure strategies s_A and s_B respectively, and event ω has occurred. In this situation, player A wins the competition if and only if $v_\omega(s_A) > v_\omega(s_B)$, and thus the utility of player A given event ω can be written as follows:

$$u_\omega^A(s_A, s_B) = \begin{cases} +1 & \text{if } v_\omega(s_A) > v_\omega(s_B) \\ 0 & \text{if } v_\omega(s_A) = v_\omega(s_B) \\ -1 & \text{if } v_\omega(s_A) < v_\omega(s_B) \end{cases}$$

Now consider a situation where players A and B play mixed strategies \mathbf{x} and \mathbf{y} respectively and event ω has occurred. The utility of player A is the probability that player A wins the competition minus the probability that player B wins the competition and can be defined as follows:

$$u_\omega^A(\mathbf{x}, \mathbf{y}) = Pr_{\substack{s_A \sim \mathbf{x} \\ s_B \sim \mathbf{y}}} [v_\omega(s_A) > v_\omega(s_B)] - Pr_{\substack{s_A \sim \mathbf{x} \\ s_B \sim \mathbf{y}}} [v_\omega(s_A) < v_\omega(s_B)]$$

Finally the overall utility of player A is $u^A(\mathbf{x}, \mathbf{y}) = \sum_\omega p_\omega u_\omega^A(\mathbf{x}, \mathbf{y})$. Since dueling game \mathcal{G} is a zero-sum game the utility of player B is the negation of the utility of player A for each ω , i.e., $u_\omega^B(\mathbf{x}, \mathbf{y}) = -u_\omega^A(\mathbf{x}, \mathbf{y})$ and thus $u^B(\mathbf{x}, \mathbf{y}) = -u^A(\mathbf{x}, \mathbf{y})$.

Definition 2.1 Minimax strategy: *Strategy \mathbf{x} of player A is minimax if $\mathbf{x} \in \operatorname{argmax}_{\mathbf{x}'} \{ \min_{\mathbf{y}} \{ u^A(\mathbf{x}', \mathbf{y}) \} \}$. Similarly, Strategy \mathbf{y} of player B is minimax if $\mathbf{y} \in \operatorname{argmax}_{\mathbf{y}'} \{ \min_{\mathbf{x}} \{ u^B(\mathbf{x}, \mathbf{y}') \} \}$.*

Based on the definition of dueling games and the fact that the set of all possible pure strategies for both players is S , we can conclude the outcome of both players in any Nash equilibrium is 0 and moreover the set of minimax strategies of both players coincide. We define the set of minimax strategies by \mathcal{M} .

Definition 2.2 Social welfare: *Consider dueling game $\mathcal{G} = (\Omega, p, S, v)$. The social welfare of pure strategy s is the expected value of this strategy over all possible events and can be written as $SW(s) = \sum_\omega p_\omega v_\omega(s)$. The social welfare of mixed strategy \mathbf{x} is $SW(\mathbf{x}) = E_{s \sim \mathbf{x}}[SW(s)]$.*

In this paper, we are interested to study the social welfare of the game in equilibria. Note that the customer locks into one of the players in long term. On the other hand, both players only try to offer the customer a better option than the other one, and thus play a minimax strategy in the competition. These cause inefficiency in the game. Here we define a new criterion to measure this inefficiency in the game.

Definition 2.3 Price of competition: *The price of competition is the ratio of the worst minimax strategy to the optimal solution which is:*

$$\frac{\min_{\mathbf{x} \in \mathcal{M}} SW(\mathbf{x})}{\max_{\mathbf{x}} SW(\mathbf{x})} = \frac{\min_{\mathbf{x} \in \mathcal{M}} SW(\mathbf{x})}{\max_{s \in S} SW(s)}.$$

Similar to the welfare maximization model, we consider the cost minimization model in which players try to beat the opponent by offering a lower cost to the user. In particular we have a cost function c , such that $c_\omega(s)$ denotes the cost of strategy s and event ω . Hence, the utility of player A would be defined as

$$u_\omega^A(\mathbf{x}, \mathbf{y}) = Pr_{\substack{s_A \sim \mathbf{x} \\ s_B \sim \mathbf{y}}} [c_\omega(s_A) < c_\omega(s_B)] - Pr_{\substack{s_A \sim \mathbf{x} \\ s_B \sim \mathbf{y}}} [c_\omega(s_A) > c_\omega(s_B)].$$

Similarly we define the social cost $SC(s) = \sum_\omega p_\omega c_\omega(s)$ for a pure strategy s and $SC(\mathbf{x}) = E_{s \sim \mathbf{x}}[SC(s)]$ for a mixed strategy \mathbf{x} . Finally the price of competition in cost minimization version is defined as

$$\frac{\max_{\mathbf{x} \in \mathcal{M}} SC(\mathbf{x})}{\min_{\mathbf{x}} SC(\mathbf{x})} = \frac{\max_{\mathbf{x} \in \mathcal{M}} SC(\mathbf{x})}{\min_{s \in S} SC(s)}.$$

2.2 Ranking duel

Ranking duel is a dueling game where $\Omega = \{1, \dots, n\}$ is the set of n webpages which can be requested by a user. In this game, the set of pure strategies S is equal to the set of all possible permutations over Ω , i.e., each player outputs an ordering of webpages for the user. We denote each pure strategy of the ranking duel by π (instead of s) where $\pi(\omega)$ is the rank of webpage ω . The valuation function v of a ranking duel can be defined based on function $f : \{1, \dots, n\} \rightarrow R^+ \cup \{0\}$ as $v_\omega(\pi) = f(\pi(\omega))$. Consider mixed strategy \mathbf{x} where \mathbf{x}_π is the probability that strategy \mathbf{x} outputs permutation π . The *social welfare* of strategy \mathbf{x} can be defined as:

$$SW(\mathbf{x}) = \sum_{\omega} \sum_{\pi} p_{\omega} \mathbf{x}_{\pi} f(\pi(\omega)). \quad (1)$$

3 Price of competition in the linear ranking duel

3.1 Welfare maximization ranking duel

In this section we give bounds for the PoC in the ranking duel when the valuation function is non-negative and linear, in other words $f(i) = c(n - i) + d$, where $c, d \geq 0$.

First we formulate the social welfare of strategy \mathbf{x} and the optimal social welfare. Without loss of generality in this section we assume $p_1 \geq p_2 \geq \dots \geq p_n$. Let $Pr_{\pi \sim \mathbf{x}}[\pi(a) = i]$ denote the probability that in a randomly drawn permutation π from strategy \mathbf{x} , the rank of webpage a is i . Similarly let $Pr_{\pi \sim \mathbf{x}}[\pi(a) < \pi(b)]$ denote the probability that in a randomly drawn permutation π from strategy \mathbf{x} , webpage a comes before webpage b .

Proposition 3.1 *In a ranking duel with valuation function f and n webpages, the social welfare of a strategy \mathbf{x} is $SW_f(\mathbf{x}) = \sum_{a=1}^n \sum_{i=1}^n p_a Pr_{\pi \sim \mathbf{x}}[\pi(a) = i] f(i)$.*

Proof. For any page a and position i we compute the probability that a is chosen and it is located at position i times $f(i)$. Hence,

$$SW(\mathbf{x}) = \sum_{a=1}^n \sum_{i=1}^n p_a Pr_{\pi \sim \mathbf{x}}[\pi(a) = i] f(i)$$

□

Let OPT be the strategy with the maximum social welfare. Hence $SW(\text{OPT})$ is formulated as follows.

Proposition 3.2 *In a ranking duel with valuation function f and n webpages, the optimal social welfare is $SW_f(\text{OPT}) = \sum_{a=1}^n p_a f(a)$.*

Proof. The optimal strategy is to sort the pages by descending order of their probability, therefore $SW_f(\text{OPT})$ is equal to the social welfare of permutation $\pi = \langle 1, 2, \dots, n \rangle$. Thus,

$$SW_f(\text{OPT}) = \sum_{a=1}^n p_a f(a).$$

□

Lemma 3.3 shows that for any minimax strategy \mathbf{x} and any linear function $f(i) = c(n - i) + d$ with $c, d \geq 0$, the PoC is no less than the case in which $f(i) = n - i$.

Lemma 3.3 *For valuation functions $f(i) = n - i$, $f'(i) = c(n - i) + d$ with $c, d \geq 0$, and any strategy \mathbf{x} , $\frac{SW_f(\mathbf{x})}{SW_f(\text{OPT})} \leq \frac{SW_{f'}(\mathbf{x})}{SW_{f'}(\text{OPT})}$.*

Proof. By Propositions 3.1 and 3.2, we have

$$\begin{aligned} \frac{SW_f(\mathbf{x})}{SW_f(\text{OPT})} &= \frac{\sum_{a=1}^n \sum_{i=1}^n p_a Pr_{\pi \sim \mathbf{x}}[\pi(a) = i](n - i)}{\sum_{a=1}^n p_a (n - a)} && \text{By multiplying the sides by } c \\ &= \frac{\sum_{a=1}^n \sum_{i=1}^n p_a Pr_{\pi \sim \mathbf{x}}[\pi(a) = i]c(n - i)}{\sum_{a=1}^n p_a c(n - a)} && \text{Since the fraction is less than 1} \\ &\leq \frac{d + \sum_{a=1}^n \sum_{i=1}^n p_a Pr_{\pi \sim \mathbf{x}}[\pi(a) = i]c(n - i)}{d + \sum_{a=1}^n p_a c(n - a)} && \text{Since } \sum_{i=1}^n p_a Pr_{\pi \sim \mathbf{x}}[\pi(a) = i] = \sum_{a=1}^n p_a = 1 \\ &= \frac{\sum_{a=1}^n \sum_{i=1}^n p_a Pr_{\pi \sim \mathbf{x}}[\pi(a) = i](c(n - i) + d)}{\sum_{a=1}^n p_a (c(n - a) + d)} \\ &= \frac{SW_{f'}(\mathbf{x})}{SW_{f'}(\text{OPT})}. \end{aligned}$$

□

Thus any lower bound for the PoC with $f(i) = n - i$, is also a lower bound for the PoC with any other linear valuation function. Therefore, from now on we assume $f(i) = n - i$, and use $SW(\mathbf{x})$ and $SW(\text{OPT})$ instead of $SW_f(\mathbf{x})$ and $SW_f(\text{OPT})$, respectively. Hence $SW(\text{OPT}) = \sum_{a=1}^n p_a (n - a)$. Now we try to compute $SW(\mathbf{x})$ from a different perspective.

Proposition 3.4 *In a ranking duel with n webpages, the social welfare of strategy \mathbf{x} is*

$$SW(\mathbf{x}) = \sum_{a=1}^n \sum_{b=a+1}^n p_a Pr_{\pi \sim \mathbf{x}}[\pi(a) < \pi(b)] + p_b Pr_{\pi \sim \mathbf{x}}[\pi(b) < \pi(a)].$$

Proof. By Proposition 3.1,

$$\begin{aligned}
\text{SW}(\mathbf{x}) &= \sum_{a=1}^n \sum_{i=1}^n p_a \Pr_{\pi \sim \mathbf{x}}[\pi(a) = i](n-i) \\
&= \sum_{a=1}^n p_a \sum_{i=1}^n \Pr_{\pi \sim \mathbf{x}}[\pi(a) = i](n-i) && \text{for each page } a \text{ at position } i \text{ in } \pi, \\
&&& \text{consider } (n-i) \text{ pages at higher positions} \\
&= \sum_{a=1}^n p_a \sum_{b=1}^n \Pr_{\pi \sim \mathbf{x}}[\pi(a) < \pi(b)] && \text{by considering each pair of pages } a \text{ and } b \text{ once} \\
&= \sum_{a=1}^n \sum_{b=a+1}^n p_a \Pr_{\pi \sim \mathbf{x}}[\pi(a) < \pi(b)] + p_b \Pr_{\pi \sim \mathbf{x}}[\pi(b) < \pi(a)].
\end{aligned}$$

□

Intuitively by Proposition 3.4 we can compute the social welfare of a strategy by comparing the ranks of every pairs of webpages. Therefore we define $h_{ab}(\mathbf{x})$ to be the amount that the pair of webpages a and b adds to the social welfare in strategy \mathbf{x} , i.e. $h_{ab}(\mathbf{x}) = p_a \Pr_{\pi \sim \mathbf{x}}[\pi(a) < \pi(b)] + p_b \Pr_{\pi \sim \mathbf{x}}[\pi(b) < \pi(a)]$. Thus we can rewrite Proposition 3.4 as $\text{SW}(\mathbf{x}) = \sum_{a=1}^n \sum_{b=a+1}^n h_{ab}(\mathbf{x})$. Hence for every strategy \mathbf{x} ,

$$\frac{\text{SW}(\mathbf{x})}{\text{SW}(\text{OPT})} = \frac{\sum_{a=1}^n \sum_{b=a+1}^n h_{ab}(\mathbf{x})}{\sum_{a=1}^n p_a(n-a)}.$$

In Lemma 3.6 we provide our main tool for bounding the price of competition in the linear ranking duel.

For proving Lemma 3.6, first we need to prove the following lemma.

Lemma 3.5 *For any three integer numbers n , a , and k such that $1 \leq a \leq n-1$ and $2 \leq k \leq n$, we have*

$$\sum_{i=0}^{k-1} \binom{a-1}{i} \binom{n-a}{k-i-1} (k-i-1) = (n-a) \binom{n-2}{k-2} \quad (2)$$

Proof. We use double counting. Consider a problem in which we have n balls numbered from 1 to n and our goal is to color the balls such that

- We have $k-2$ black balls, 1 red ball, and $n-(k-2)-1$ uncolored balls.
- a -th ball is uncolored.
- Index of the red ball is higher than a .

We calculate the number of different ways that we can color the balls. One way to calculate this is to select $k-1$ balls to color first, and then color $k-2$ of them with black and one of them which has an index higher than a with red. The number of such colorings can be calculated as follows

$$\sum_{i=0}^{k-1} \binom{a-1}{i} \binom{n-a}{k-i-1} (k-i-1),$$

which is equal to the left side of Equation (3.5). The other way to count the number of valid colorings is to first color one ball with rank higher than a with red, and then color $k-2$ balls out

of all balls except the a -th and the red ball. We have $n - a$ choices for coloring the red ball and $\binom{n-2}{k-2}$ choices for coloring the black ones. Thus we have $(n - a)\binom{n-2}{k-2}$ different ways, which is equal to the right side of Equation (3.5). Therefore, both sides of Equation (3.5) are equal the number of different valid colorings. \square

Lemma 3.6 *Given a strategy \mathbf{x} , if there exist an integer k such that $2 \leq k \leq n$ and for all k different indices $i_1 < i_2 < \dots < i_k$,*

$$\frac{\sum_{a=1}^k \sum_{b=a+1}^k h_{i_a i_b}(\mathbf{x})}{\sum_{a=1}^k p_{i_a}(k - a)} \geq \alpha,$$

then $\frac{SW(\mathbf{x})}{SW(OPT)} \geq \alpha$.

Proof. We compute the summation of

$$\sum_{a=1}^k \sum_{b=a+1}^k p_{i_a} Pr_{\pi \sim \mathbf{x}}[\pi(i_a) < \pi(i_b)] + p_{i_b} Pr_{\pi \sim \mathbf{x}}[\pi(i_b) < \pi(i_a)] \geq \alpha \left(\sum_{a=1}^k p_{i_a}(k - a) \right)$$

for all $\binom{n}{k}$ possible indices $1 \leq i_1 < \dots < i_k \leq n$, therefore we have:

$$\sum_{1 \leq i_1 < \dots < i_k \leq n} \sum_{a=1}^k \sum_{b=a+1}^k p_{i_a} Pr_{\pi \sim \mathbf{x}}[\pi(i_a) < \pi(i_b)] + p_{i_b} Pr_{\pi \sim \mathbf{x}}[\pi(i_b) < \pi(i_a)] \geq \quad (3)$$

$$\alpha \left(\sum_{1 \leq i_1 < \dots < i_k \leq n} \sum_{a=1}^k p_{i_a}(k - a) \right)$$

For each $1 \leq a < b \leq k$, $Pr_{\pi \sim \mathbf{x}}[\pi(i_a) < \pi(i_b)] + p_{i_b} Pr_{\pi \sim \mathbf{x}}[\pi(i_b) < \pi(i_a)]$ appears $\binom{n-2}{k-2}$ times in the left side, hence

$$\begin{aligned} & \sum_{1 \leq i_1 < i_2 < i_3 < \dots < i_k \leq n} \sum_{a=1}^k \sum_{b=a+1}^k p_{i_a} Pr_{\pi \sim \mathbf{x}}[\pi(i_a) < \pi(i_b)] + p_{i_b} Pr_{\pi \sim \mathbf{x}}[\pi(i_b) < \pi(i_a)] \\ &= \sum_{a=1}^n \sum_{b=a+1}^n \binom{n-2}{k-2} (p_a Pr_{\pi \sim \mathbf{x}}[\pi(a) < \pi(b)] + p_b Pr_{\pi \sim \mathbf{x}}[\pi(b) < \pi(a)]). \end{aligned}$$

Moreover, the coefficient of p_a in the right side of Equation (3) is equal to $\sum_{b=0}^{k-2} \binom{a-1}{b} \binom{n-a}{k-1-b} (k-b)$, Therefore

$$\begin{aligned} & \sum_{a=1}^n \sum_{b=a+1}^n \binom{n-2}{k-2} (p_a Pr_{\pi \sim \mathbf{x}}[\pi(a) < \pi(b)] + p_b Pr_{\pi \sim \mathbf{x}}[\pi(b) < \pi(a)]) \geq \\ & \alpha \left(\sum_{a=1}^n p_a \sum_{b=0}^{k-2} \binom{a-1}{b} \binom{n-a}{k-1-b} (k-b) \right) \end{aligned}$$

Lemma 3.5 states that the right side is equal to $\alpha(\sum_{a=1}^n p_a(n-a)\binom{n-2}{k-2})$. Thus, by dividing both sides by $\binom{n-2}{k-2}$ we have

$$\sum_{a=1}^n \sum_{b=a+1}^n p_a Pr_{\pi \sim \mathbf{x}}[\pi(a) < \pi(b)] + p_b Pr_{\pi \sim \mathbf{x}}[\pi(b) < \pi(a)] \geq \alpha \left(\sum_{a=1}^n p_a(n-a) \right)$$

which concludes $\frac{SW(\mathbf{x})}{SW(OPT)} \geq \alpha$. \square

Now our goal is to provide a lower bound for α when \mathbf{x} is a minimax strategy. In order to do that, first we provide some structural properties of the minimax strategies. Leveraging these properties we write a mathematical program with k variables p_a and $\binom{k}{2}$ variables h_{ab} . Finally, we provide a factor-revealing linear program to obtain a close lower bound for α in the corresponding mathematical program.

In Lemmas 3.7, 3.8, 3.10, and Proposition 3.9 we provide the structural properties of the minimax strategies.

Lemma 3.7 *Let \mathbf{x} be a minimax strategy and a and b be two webpages such that $p_a \geq p_b$. Let π_{ba} be any permutation in the support of \mathbf{x} in which b precedes a . Let $i < j$ be the respective position of a and b in π_{ba} , then strategy \mathbf{x} must satisfy,*

$$Pr_{\pi \sim \mathbf{x}}[i < \pi(b) \leq j] + Pr_{\pi \sim \mathbf{x}}[i \leq \pi(b) < j] \geq \frac{p_a}{p_b}(Pr_{\pi \sim \mathbf{x}}[i < \pi(a) \leq j] + Pr_{\pi \sim \mathbf{x}}[i \leq \pi(a) < j]).$$

Proof. Let π_{ab} be a permutation which is constructed from π_{ba} by swapping a and b . This means $\pi_{ab}(a) = i$ and $\pi_{ab}(b) = j$. Consider strategy $\mathbf{x}' = \mathbf{x} + \epsilon_{\pi_{ab}} - \epsilon_{\pi_{ba}}$ which is produced from \mathbf{x} by increasing the probability of π_{ab} by ϵ and decreasing the probability of π_{ba} by ϵ . Since \mathbf{x} is a minmax strategy, we have $u^A(\mathbf{x}, \mathbf{x}') \geq 0$.

As mentioned in Section 2.1, $u_w^A(\mathbf{x}, \mathbf{x}') = \sum_{\omega} u_{\omega}^A(x, \mathbf{x}')$, where $u_{\omega}^A(\mathbf{x}, \mathbf{x}')$ is the payoff of the game when webpage ω is searched.

$$u(\mathbf{x}, \mathbf{x}') = \sum_{\omega=1}^n p_{\omega} u_{\omega}^A(\mathbf{x}, \mathbf{x}') \quad (4)$$

Since, both players act the same for all pages except a and b , we have

$$\forall 1 \leq \omega \leq n, \omega \neq a, \omega \neq b \quad u_{\omega}^A(\mathbf{x}, \mathbf{x}') = 0. \quad (5)$$

Therefore

$$u^A(\mathbf{x}, \mathbf{x}') = p_a u_a^A(\mathbf{x}, \mathbf{x}') + p_b u_b^A(\mathbf{x}, \mathbf{x}'). \quad (6)$$

Let \mathbf{x}'' be a new strategy such that $\mathbf{x}''_{\pi_{ba}} = \mathbf{x}_{\pi_{ba}} - \epsilon$, and for every other permutation π , $\mathbf{x}''_{\pi} = \frac{1}{1-\epsilon} \mathbf{x}_{\pi}$. In fact strategy \mathbf{x} plays strategy \mathbf{x}'' with probability $1 - \epsilon$ and plays permutation π_{ba} with probability ϵ . Also strategy \mathbf{x}' plays \mathbf{x}'' with probability $1 - \epsilon$ and plays permutation π_{ab} with probability ϵ . Therefore we can compute $u_a^A(\mathbf{x}, \mathbf{x}')$ as follows

$$\begin{aligned} u_a^A(\mathbf{x}, \mathbf{x}') &= (1 - \epsilon)^2 (Pr_{\pi_1 \sim \mathbf{x}'', \pi_2 \sim \mathbf{x}''}[\pi_1(a) < \pi_2(a)] - Pr_{\pi_1 \sim \mathbf{x}'', \pi_2 \sim \mathbf{x}''}[\pi_1(a) > \pi_2(a)]) \\ &\quad \text{The case that both strategies play } \mathbf{x}'' \\ &+ \epsilon(1 - \epsilon)(Pr_{\pi_2 \sim \mathbf{x}''}[\pi_{ba}(a) < \pi_2(a)] - Pr_{\pi_2 \sim \mathbf{x}''}[\pi_{ba}(a) > \pi_2(a)]) \\ &\quad \text{The case that } \mathbf{x} \text{ plays } \pi_{ba} \text{ and } \mathbf{x}' \text{ plays } \mathbf{x}'' \\ &+ \epsilon(1 - \epsilon)(Pr_{\pi_1 \sim \mathbf{x}''}[\pi_1(a) < \pi_{ab}(a)] - Pr_{\pi_1 \sim \mathbf{x}''}[\pi_1(a) > \pi_{ab}(a)]) \\ &\quad \text{The case that } \mathbf{x} \text{ plays } \mathbf{x}'' \text{ and } \mathbf{x}' \text{ plays } \pi_{ab} \\ &+ \epsilon^2 (Pr[\pi_{ba}(a) < \pi_{ab}(a)] - Pr[\pi_{ba}(a) > \pi_{ab}(a)]) \\ &\quad \text{The case that } \mathbf{x} \text{ plays } \pi_{ba} \text{ and } \mathbf{x}' \text{ plays } \pi_{ab} \end{aligned}$$

Consider the first term of the above equation. Since both permutations π_1 and π_2 are drawn from the same distribution \mathbf{x}'' , this term is zero. On the other hand, recall that $\pi_{ab}(a) = \pi_{ba}(b) = i$ and

$\pi_{ab}(b) = \pi_{ba}(a) = j$. Thus $Pr[\pi_{ba}(a) < \pi_{ab}(a)] - Pr[\pi_{ba}(a) > \pi_{ab}(a)] = Pr[j < i] - Pr[j > i] = -1$, hence

$$\begin{aligned} u_a^A(\mathbf{x}, \mathbf{x}') &= \epsilon(1 - \epsilon)(Pr_{\pi_2 \sim \mathbf{x}''}[j < \pi_2(a)] - Pr_{\pi_2 \sim \mathbf{x}''}[j > \pi_2(a)]) \\ &+ \epsilon(1 - \epsilon)(Pr_{\pi_1 \sim \mathbf{x}''}[\pi_1(a) < i] - Pr_{\pi_1 \sim \mathbf{x}''}[\pi_1(a) > i]) \\ &- \epsilon^2 \end{aligned} \quad (7)$$

Similarly we have

$$\begin{aligned} u_b^A(\mathbf{x}, \mathbf{x}') &= \epsilon(1 - \epsilon)(Pr_{\pi_2 \sim \mathbf{x}''}[i < \pi_2(b)] - Pr_{\pi_2 \sim \mathbf{x}''}[i > \pi_2(b)]) \\ &+ \epsilon(1 - \epsilon)(Pr_{\pi_1 \sim \mathbf{x}''}[\pi_1(b) < j] - Pr_{\pi_1 \sim \mathbf{x}''}[\pi_1(b) > j]) \\ &+ \epsilon^2 \end{aligned} \quad (8)$$

$u^A(\mathbf{x}, \mathbf{x}') \geq 0$ thus by Equation (6) $p_a u_a^A(\mathbf{x}, \mathbf{x}') + p_b u_b^A(\mathbf{x}, \mathbf{x}') \geq 0$. Hence using Equations (7) and (8) we have

$$\begin{aligned} &p_a \epsilon(1 - \epsilon)(Pr_{\pi_2 \sim \mathbf{x}''}[j < \pi_2(a)] - Pr_{\pi_2 \sim \mathbf{x}''}[j > \pi_2(a)]) \\ &+ p_a \epsilon(1 - \epsilon)(Pr_{\pi_1 \sim \mathbf{x}''}[\pi_1(a) < i] - Pr_{\pi_1 \sim \mathbf{x}''}[\pi_1(a) > i]) - p_a \epsilon^2 \\ &+ p_b \epsilon(1 - \epsilon)(Pr_{\pi_2 \sim \mathbf{x}''}[i < \pi_2(b)] - Pr_{\pi_2 \sim \mathbf{x}''}[i > \pi_2(b)]) \\ &+ p_b \epsilon(1 - \epsilon)(Pr_{\pi_1 \sim \mathbf{x}''}[\pi_1(b) < j] - Pr_{\pi_1 \sim \mathbf{x}''}[\pi_1(b) > j]) + p_b \epsilon^2 \geq 0 \end{aligned}$$

Note that $p_a \geq p_b$ which means $-p_a \epsilon^2 + p_b \epsilon^2 \leq 0$. Thus we conclude:

$$\begin{aligned} &p_a(Pr_{\pi_2 \sim \mathbf{x}''}[j < \pi_2(a)] - Pr_{\pi_2 \sim \mathbf{x}''}[j > \pi_2(a)]) \\ &+ p_a(Pr_{\pi_1 \sim \mathbf{x}''}[\pi_1(a) < i] - Pr_{\pi_1 \sim \mathbf{x}''}[\pi_1(a) > i]) \\ &+ p_b(Pr_{\pi_2 \sim \mathbf{x}''}[i < \pi_2(b)] - Pr_{\pi_2 \sim \mathbf{x}''}[i > \pi_2(b)]) \\ &+ p_b(Pr_{\pi_1 \sim \mathbf{x}''}[\pi_1(b) < j] - Pr_{\pi_1 \sim \mathbf{x}''}[\pi_1(b) > j]) \geq 0 \end{aligned} \quad (9)$$

Note that as ϵ approaches zero, strategy \mathbf{x}'' approaches \mathbf{x} . Hence, we write Equation (9) as follows:

$$\begin{aligned} &p_a(Pr_{\pi \sim \mathbf{x}}[j < \pi(a)] - Pr_{\pi \sim \mathbf{x}}[j > \pi(a)]) \\ &+ p_a(Pr_{\pi \sim \mathbf{x}}[\pi(a) < i] - Pr_{\pi \sim \mathbf{x}}[\pi(a) > i]) \\ &+ p_b(Pr_{\pi \sim \mathbf{x}}[i < \pi(b)] - Pr_{\pi \sim \mathbf{x}}[i > \pi(b)]) \\ &+ p_b(Pr_{\pi \sim \mathbf{x}}[\pi(b) < j] - Pr_{\pi \sim \mathbf{x}}[\pi(b) > j]) \geq 0 \end{aligned} \quad (10)$$

Rearranging the terms we have

$$Pr_{\pi \sim \mathbf{x}}[i < \pi(b) \leq j] + Pr_{\pi \sim \mathbf{x}}[i \leq \pi(b) < j] \geq \frac{p_a}{p_b}(Pr_{\pi \sim \mathbf{x}}[i < \pi(a) \leq j] + Pr_{\pi \sim \mathbf{x}}[i \leq \pi(a) < j]).$$

□

Intuitively Lemma 3.7 shows that if $p_a \geq p_b$ and there is a permutation in which b comes before a , then the probability that \mathbf{x} ranks b in interval $[i, j]$ (counting the non-endpoint elements twice) is greater than the probability that \mathbf{x} ranks a in this interval by a factor of $\frac{p_a}{p_b}$. Otherwise, by swapping the rank of a and b we can achieve a strategy that beats \mathbf{x} .

Lemma 3.8 *Let \mathbf{x} be a minimax strategy and \mathbf{x}_π be the probability that strategy \mathbf{x} plays permutation π . For every pair of webpages a and b with $p_a \geq p_b$, we have*

$$Pr_{\pi \sim \mathbf{x}}[\pi(a) < \pi(b)] \geq (\frac{p_a}{2p_b} - 1)Pr_{\pi \sim \mathbf{x}}[\pi(b) < \pi(a)]. \quad (11)$$

Proof. First we find the set Π using Algorithm 1. Let N_{ab} be the set of all strategies in \mathbf{x} in which a comes before b . Similarly let N_{ba} be the set of strategies in \mathbf{x} in which b comes before a . In Algorithm 1, initially we consider a set S^* to be N_{ba} . Then, at each step we add permutation $\pi \in S^*$ with the rightmost a to Π , and remove all permutations π' such that the interval $[\pi'(b), \pi'(a)]$ overlaps the interval $[\pi(b), \pi(a)]$ from S^* . We repeat this process until all permutations are removed from S^* .

Algorithm 1

input: \mathbf{x}, a, b

- 1: $S^* \leftarrow N_{ba}$.
 - 2: $\Pi \leftarrow \emptyset$.
 - 3: **while** $S^* \neq \emptyset$ **do**
 - 4: Let $\pi \in S^*$ be the permutation with the rightmost $\pi(a)$ among all permutations in S^* .
 - 5: $\Pi \leftarrow \Pi \cup \{\pi\}$.
 - 6: $S^* \leftarrow \{\pi' \in S^* \mid [\pi'(b), \pi'(a)] \cap [\pi(b), \pi(a)] = \emptyset\}$.
 - 7: **end while**
 - 8: **return** Π
-

Afterwards for each $\pi \in \Pi$, we apply Lemma 3.7 and add all these inequalities to reach the following inequality:

$$\begin{aligned} & \sum_{\pi \in \Pi} Pr_{\pi' \sim \mathbf{x}}[\pi(b) < \pi'(b) \leq \pi(a)] + Pr_{\pi' \sim \mathbf{x}}[\pi(b) \leq \pi'(b) < \pi(a)] \\ & \geq \sum_{\pi \in \Pi} \frac{p_a}{p_b} (Pr_{\pi' \sim \mathbf{x}}[\pi(b) < \pi'(a) \leq \pi(a)] + Pr_{\pi' \sim \mathbf{x}}[\pi(b) \leq \pi'(a) < \pi(b)]). \end{aligned} \quad (12)$$

We can partition each term of Inequality (12) into two terms such that in one of them $\pi'(a) > \pi'(b)$ and in the other one $\pi'(b) > \pi'(a)$. Thus we can rewrite Inequality (12) as follows

$$\begin{aligned} & \sum_{\pi \in \Pi} Pr_{\pi' \sim \mathbf{x}}[\pi'(a) < \pi'(b) \wedge \pi(b) < \pi'(b) \leq \pi(a)] \\ & + Pr_{\pi' \sim \mathbf{x}}[\pi'(b) < \pi'(a) \wedge \pi(b) < \pi'(b) \leq \pi(a)] \\ & + Pr_{\pi' \sim \mathbf{x}}[\pi'(a) < \pi'(b) \wedge \pi(b) \leq \pi'(b) < \pi(a)] \\ & + Pr_{\pi' \sim \mathbf{x}}[\pi'(b) < \pi'(a) \wedge \pi(b) \leq \pi'(b) < \pi(a)] \\ & \geq \frac{p_a}{p_b} \sum_{\pi \in \Pi} Pr_{\pi' \sim \mathbf{x}}[\pi'(a) < \pi'(b) \wedge \pi(b) < \pi'(a) \leq \pi(a)] \\ & + Pr_{\pi' \sim \mathbf{x}}[\pi'(b) < \pi'(a) \wedge \pi(b) < \pi'(a) \leq \pi(a)] \\ & + Pr_{\pi' \sim \mathbf{x}}[\pi'(a) < \pi'(b) \wedge \pi(b) \leq \pi'(a) < \pi(b)] \\ & + Pr_{\pi' \sim \mathbf{x}}[\pi'(b) < \pi'(a) \wedge \pi(b) \leq \pi'(a) < \pi(b)]. \end{aligned} \quad (13)$$

Now for the sake of convenience, for webpage $c \in \{a, b\}$, we define R_c^{ab} and R_c^{ba} as follows.

$$R_c^{ab} = \sum_{\pi \in \Pi} Pr_{\pi' \sim \mathbf{x}}[\pi'(a) < \pi'(b) \wedge \pi(b) < \pi'(c) \leq \pi(a)] + Pr_{\pi' \sim \mathbf{x}}[\pi'(a) < \pi'(b) \wedge \pi(b) \leq \pi'(c) < \pi(a)].$$

$$R_c^{ba} = \sum_{\pi \in \Pi} Pr_{\pi' \sim \mathbf{x}}[\pi'(b) < \pi'(a) \wedge \pi(b) < \pi'(c) \leq \pi(a)] + Pr_{\pi' \sim \mathbf{x}}[\pi'(b) < \pi'(a) \wedge \pi(b) \leq \pi'(c) < \pi(a)].$$

Thus we can rewrite Inequality (13) based on the above definitions, and conclude

$$R_b^{ab} + R_b^{ba} \geq \frac{p_a}{p_b} (R_a^{ab} + R_a^{ba}). \quad (14)$$

Note that by the definition of R , for each permutation $\pi' \in \Pi$ a randomly drawn permutation π' from \mathbf{x} and a webpage $c \in \{a, b\}$, we add 2 to R_c if $\pi'(c)$ is between the rank of a and b in π , and we add 1 if $\pi'(c)$ is equal to the rank of either a or b in π . More formally we define $r_c(\pi, \pi')$ as follows

$$r_c(\pi', \pi) = \begin{cases} 2 & \text{if } \pi(a) < \pi'(c) < \pi(b) \text{ or } \pi(b) < \pi'(c) < \pi(a) \\ 1 & \text{if } \pi'(c) = \pi(a) \text{ or } \pi'(c) = \pi(b) \\ 0 & \text{if otherwise} \end{cases}$$

Thus we can write R_c^{ab} and R_c^{ba} as follows

$$\begin{aligned} R_c^{ab} &= \sum_{\pi \in \Pi} \sum_{\pi' \in N_{ab}} \mathbf{x}_{\pi'} r_c(\pi', \pi). \\ R_c^{ba} &= \sum_{\pi \in \Pi} \sum_{\pi' \in N_{ba}} \mathbf{x}_{\pi'} r_c(\pi', \pi). \end{aligned} \quad (15)$$

Regarding the construction of Π , for any rank i , there is at most one permutation $\pi \in \Pi$ such that i is between $\pi(a)$ and $\pi(b)$. Thus for a permutation π' , there is at most one permutation $\pi \in \Pi$ such that $r_b(\pi', \pi)$ is non-zero. Moreover $r_b(\pi', \pi) \leq 2$. Thus for any permutation π we have

$$\sum_{\pi \in \Pi} r_b(\pi', \pi) \leq 2. \quad (16)$$

As a result by Equation (15) and Inequality (16), we have

$$R_b^{ba} = \sum_{\pi' \in N_{ba}} \sum_{\pi \in \Pi} \mathbf{x}_{\pi'} r_b(\pi', \pi) \leq \sum_{\pi' \in N_{ba}} 2\mathbf{x}_{\pi'} \quad (17)$$

$$R_b^{ab} = \sum_{\pi' \in N_{ab}} \sum_{\pi \in \Pi} \mathbf{x}_{\pi'} r_b(\pi', \pi) \leq \sum_{\pi' \in N_{ab}} 2\mathbf{x}_{\pi'} \quad (18)$$

On the other hand for every rank i if there is a permutation $\pi' \in N_{ba}$ with $\pi'(a) = i$, then there is exactly one permutation $\pi \in \Pi$ such that $i \in [\pi(b), \pi(a)]$ and hence $r_a(\pi', \pi) \geq 1$. Thus,

$$R_a^{ba} = \sum_{\pi' \in N_{ba}} \sum_{\pi \in \Pi} \mathbf{x}_{\pi'} r_a(\pi', \pi) \geq \sum_{\pi' \in N_{ba}} \mathbf{x}_{\pi'} \quad (19)$$

By Inequalities (17) and (19) we have

$$R_b^{ba} \leq 2R_a^{ba} \quad (20)$$

Now we can prove the lemma. By Equation (14),

$$R_b^{ab} + R_b^{ba} \geq \frac{p_a}{p_b}(R_a^{ab} + R_a^{ba}) \quad \text{Since } R_a^{ab} \geq 0$$

$$R_b^{ab} + R_b^{ba} \geq \frac{p_a}{p_b} R_a^{ba} \quad \text{By Inequality (20)}$$

$$R_b^{ab} + 2R_a^{ba} \geq \frac{p_a}{p_b} R_a^{ba}$$

$$R_b^{ab} \geq \left(\frac{p_a}{p_b} - 2\right) R_a^{ba} \quad \text{By Inequality (18)}$$

$$\sum_{\pi' \in N_{ab}} 2\mathbf{x}_{\pi'} \geq \left(\frac{p_a}{p_b} - 2\right) R_a^{ba} \quad \text{By Inequality (19)}$$

$$\sum_{\pi' \in N_{ab}} 2\mathbf{x}_{\pi'} \geq \left(\frac{p_a}{p_b} - 2\right) \sum_{\pi' \in N_{ba}} \mathbf{x}_{\pi'}$$

Since $\sum_{\pi' \in N_{ab}} \mathbf{x}_{\pi'} = Pr_{\pi \sim \mathbf{x}}[\pi(a) < \pi(b)]$ and $\sum_{\pi' \in N_{ba}} \mathbf{x}_{\pi'} = Pr_{\pi \sim \mathbf{x}}[\pi(b) < \pi(a)]$,

$$Pr_{\pi \sim \mathbf{x}}[\pi(a) < \pi(b)] \geq \left(\frac{p_a}{2p_b} - 1\right) Pr_{\pi \sim \mathbf{x}}[\pi(b) < \pi(a)].$$

□

Briefly, in the proof of Lemma 3.8 we propose an algorithm to find a set of permutations Π in \mathbf{x} , such that 1) for each $\pi \in \Pi$, b comes before a , 2) for each permutation π' in \mathbf{x} in which b comes before a , there is a permutation $\pi \in \Pi$, such that $\pi(b) \leq \pi'(a) \leq \pi(a)$, and 3) the interval of the ranks of b and a are distinct, i.e. for two permutations $\pi, \pi' \in \Pi$, $[\pi(b), \pi(a)] \cap [\pi'(b), \pi'(a)] = \emptyset$. We apply the inequality in Lemma 3.7 for all permutations in Π to achieve Lemma 3.8.

In Proposition 3.9 and Lemma 3.10 we provide lower bounds for $h_{ab}(\mathbf{x})$ when \mathbf{x} is a minimax strategy. Hence we can use these lower bounds in the proposed mathematical program to achieve a lower bound for the PoC.

Proposition 3.9 *For minimax strategy \mathbf{x} and webpages a and b such that $p_a \geq p_b$, $h_{ab}(\mathbf{x}) \geq p_b$.*

Proof. Since $p_a \geq p_b$ and $Pr_{\pi \sim \mathbf{x}}[\pi(a) < \pi(b)] + Pr_{\pi \sim \mathbf{x}}[\pi(b) < \pi(a)] = 1$, we have:

$$\begin{aligned} p_a Pr_{\pi \sim \mathbf{x}}[\pi(a) < \pi(b)] + p_b Pr_{\pi \sim \mathbf{x}}[\pi(b) < \pi(a)] &\geq p_b Pr_{\pi \sim \mathbf{x}}[\pi(a) < \pi(b)] + p_b Pr_{\pi \sim \mathbf{x}}[\pi(b) < \pi(a)] \\ &= p_b \end{aligned}$$

□

Lemma 3.10 *For minimax strategy \mathbf{x} and webpages a and b such that $p_a \geq p_b$, $h_{ab}(\mathbf{x}) \geq p_a - 2p_b + \frac{2p_b^2}{p_a}$.*

Proof. First we claim that $Pr_{\pi \sim \mathbf{x}}[\pi(a) < \pi(b)] \geq 1 - \frac{2p_b}{p_a}$. By Lemma 3.8,

$$\begin{aligned}
Pr_{\pi \sim \mathbf{x}}[\pi(a) < \pi(b)] &\geq \left(\frac{p_a}{2p_b} - 1\right) Pr_{\pi \sim \mathbf{x}}[\pi(b) < \pi(a)] && \text{dividing the sides by } Pr_{\pi \sim \mathbf{x}}[\pi(b) < \pi(a)] \\
\Rightarrow \frac{Pr_{\pi \sim \mathbf{x}}[\pi(a) < \pi(b)]}{Pr_{\pi \sim \mathbf{x}}[\pi(b) < \pi(a)]} &\geq \left(\frac{p_a}{2p_b} - 1\right) \\
\Rightarrow \frac{1}{Pr_{\pi \sim \mathbf{x}}[\pi(b) < \pi(a)]} &\geq \frac{p_a}{2p_b} \\
\Rightarrow Pr_{\pi \sim \mathbf{x}}[\pi(b) < \pi(a)] &\leq \frac{2p_b}{p_a} && \text{since } Pr_{\pi \sim \mathbf{x}}[\pi(b) < \pi(a)] = 1 - Pr_{\pi \sim \mathbf{x}}[\pi(a) < \pi(b)] \\
\Rightarrow Pr_{\pi \sim \mathbf{x}}[\pi(a) < \pi(b)] &\geq 1 - \frac{2p_b}{p_a} && (21)
\end{aligned}$$

Now we use Inequality (21) and the fact that $p_a \geq p_b$ as follows

$$\begin{aligned}
&p_a Pr_{\pi \sim \mathbf{x}}[\pi(a) < \pi(b)] + p_b Pr_{\pi \sim \mathbf{x}}[\pi(b) < \pi(a)] \\
&= p_a Pr_{\pi \sim \mathbf{x}}[\pi(a) < \pi(b)] + p_b(1 - Pr_{\pi \sim \mathbf{x}}[\pi(a) < \pi(b)]) \\
&= p_b + (p_a - p_b) Pr_{\pi \sim \mathbf{x}}[\pi(a) < \pi(b)] && \text{By Inequality (21)} \\
&\geq p_b + (p_a - p_b)\left(1 - \frac{2p_b}{p_a}\right) = p_a - 2p_b + \frac{2p_b^2}{p_a}
\end{aligned}$$

□

Leveraging the properties of the minimax strategies we write MP 22. In MP 22, Constraints 24 and 25 force p_a 's to satisfy the probability constraints. Using Proposition 3.9, Constraint 26 forces h_{ab} to be no less than p_b and due to Lemma 3.10, Constraint 26 forces h_{ab} to be no less than $p_a - 2p_b + \frac{2p_b^2}{p_a}$. By Lemma 3.6, α in Constraint 23 gives a lower bound for the PoC.

$$\text{minimize } \alpha \tag{22}$$

$$\text{subject to } \alpha = \frac{\sum_{a=1}^k \sum_{b=a+1}^k h_{ab}}{\sum_{a=1}^k p_a(k-a)} \tag{23}$$

$$p_a \geq 0 \quad \forall 1 \leq a \leq k \tag{24}$$

$$\sum_{1 \leq a \leq k} p_a \leq 1 \tag{25}$$

$$h_{ab} \geq p_b \quad \forall 1 \leq a < b \leq k \tag{26}$$

$$h_{ab} \geq p_a - 2p_b + \frac{2p_b^2}{p_a} \quad \forall 1 \leq a < b \leq k \tag{27}$$

For each k , let α_k be the optimal value of the objective function in MP 22.

Lemma 3.11 α_k is a lower bound for the PoC of the linear ranking duel where $n \geq k$.

Proof. As we have in Lemma 3.6, in a strategy \mathbf{x} , if for every k indices $i_1 < i_2 < \dots < i_k$ we have

$$\frac{\sum_{a=1}^k \sum_{b=a+1}^k p_{i_a} Pr_{\pi \sim \mathbf{x}}[\pi(i_a) < \pi(i_b)] + p_{i_b} Pr_{\pi \sim \mathbf{x}}[\pi(i_b) < \pi(i_a)]}{\sum_{a=1}^k p_{i_a}(k-a)} \geq \alpha,$$

then $\frac{SW(\mathbf{x})}{SW(\text{OPT})} \geq \alpha$.

Now consider k indices which minimizes this fraction and for $1 \leq i_a \leq k$, let p_a be the probability of that webpage and for $1 \leq i_a < i_b \leq k$ let h_{ab} be $p_a Pr_{\pi \sim \mathbf{x}}[\pi(i_a) < \pi(i_b)] + p_b Pr_{\pi \sim \mathbf{x}}[\pi(i_b) < \pi(i_a)]$. Constraint 24 forces the probabilities to be positive and Constraint 25 forces the sum of probabilities to be less than 1. Since we are considering minmax strategies, we use Proposition 3.9 and Lemma 3.10 to give lower bounds in Constraints 26 and 27 for h_{ab} . Therefore any minmax strategy in a ranking duel is a feasible solution for MP 22. Thus α is a lower bound for the PoC. \square

In Theorem 3.13 we formally prove $\alpha_{10} \geq 0.612$, which results in $PoC \geq 0.612$ for any ranking duel with $n \geq 10$ webpages. Moreover, we write a computer program to find α_k for $2 \leq k \leq 100$ (see Figure 1).

In order to prove Theorem 3.13, first we should prove the following lemma.

Lemma 3.12 For $0 \leq p_b \leq p_a \leq 1$, $\max\{p_b, p_a - 2p_b + \frac{2p_b^2}{p_1}\} \geq \max\{p_b, p_a - 2p_b, \frac{(p_a - p_b)}{1.208}, \frac{(2p_a - p_b)}{3.2}\}$.

Proof. If $p_b = 0$ the inequality holds, thus we assume $p_b > 0$. Let $z = \frac{p_a}{p_b}$. We multiply each side by $\frac{1}{p_b}$. Thus we should prove

$$\begin{aligned} \max\left\{\frac{1}{p_b}, \frac{1}{p_b}(p_a - 2p_b) + \frac{1}{p_b} \frac{2p_b^2}{p_a}\right\} &\geq \max\left\{\frac{1}{p_b}, \frac{1}{p_b}(p_a - 2p_b), \frac{1}{p_b} \frac{p_a - p_b}{1.208}, \frac{1}{p_b} \frac{2p_a - p_b}{3.2}\right\} \\ \Rightarrow \max\left\{1, z - 2 + \frac{2}{z}\right\} &\geq \max\left\{1, z - 2, \frac{z - 1}{1.208}, \frac{2z - 1}{3.2}\right\} \end{aligned}$$

$z - 2 + \frac{2}{z} \geq z - 2$, thus it is sufficient to prove $z - 2 + \frac{2}{z} \geq \frac{z - 1}{1.208}$ and $z - 2 + \frac{2}{z} \geq \frac{2z - 1}{3.2}$. Since $(2 \times 1.208 - 1)^2 - 8(1.208 - 1) \times 1.208 > 0$, $(1.208 - 1)z^2 - z(2 \times 1.208 - 1) + 2 \times 1.208 > 0$. Thus by dividing the terms by $1.208z$ we have

$$\begin{aligned} z - 2 + \frac{2}{z} - \frac{z - 1}{1.208} &> 0 \\ \Rightarrow z - 2 + \frac{2}{z} &> \frac{z - 1}{1.208}. \end{aligned}$$

Moreover, since $(2 \times 3.2 - 1)^2 - 8(3.2 - 2) \times 3.2 > 0$, $(3.2 - 2)z^3 - z(2 \times 3.2 - 1) + 2 \times 3.2 > 0$. Thus by dividing the terms by $3.2z$ we have

$$\begin{aligned} z - 2 + \frac{2}{z} - \frac{2z - 1}{3.2} &> 0 \\ \Rightarrow z - 2 + \frac{2}{z} &> \frac{2z - 1}{3.2}. \end{aligned}$$

\square

Theorem 3.13 For a linear ranking duel with $n \geq 10$ webpages, $PoC \geq 0.612$.

Proof. First we try to replace Constraint 27 with some linear constraints. More precisely we replace $\max\{p_b, p_a - 2p_b + \frac{2p_b^2}{p_1}\}$ by the maximum of four linear terms. Lemma 3.12 shows we can

change the MP 22 to achieve the following program.

$$\text{minimize } \alpha \tag{28}$$

$$\text{subject to } \alpha = \frac{\sum_{a=1}^k \sum_{b=a+1}^k h_{ab}}{\sum_{a=1}^k p_a(k-a)} \tag{29}$$

$$p_a \geq 0 \quad \forall 1 \leq a \leq k \tag{30}$$

$$\sum_{1 \leq a \leq k} p_a \leq 1 \tag{31}$$

$$h_{ab} \geq p_b \quad \forall 1 \leq a < b \leq k \tag{32}$$

$$h_{ab} \geq p_a - 2p_b \quad \forall 1 \leq a < b \leq k \tag{33}$$

$$h_{ab} \geq \frac{p_a - p_b}{1.208} \quad \forall 1 \leq a < b \leq k \tag{34}$$

$$h_{ab} \geq \frac{2p_a - p_b}{3.2} \quad \forall 1 \leq a < b \leq k \tag{35}$$

Still Constraint 29 is not linear. We scale the probabilities such that $\sum_{a=1}^k p_a(k-a)$ becomes equal to 1, hence we can have a linear constraint instead of Constraint 29. Let $(\alpha, V = \langle v_{12}, v_{13}, \dots, v_{k-1k} \rangle, P = \langle p_1, p_2, \dots, p_k \rangle)$ be a feasible solution for MP 29. We provide a feasible solution with variables $(\alpha', V' = \langle v'_{12}, v'_{13}, \dots, v'_{k-1k} \rangle, P' = \langle p'_1, p'_2, \dots, p'_k \rangle)$ such that $\alpha' = \alpha$. Let $c = \frac{1}{\sum_{a=1}^k p_a(k-a)}$. Let $p'_i = cp_i$ and $h'_{ab} = ch_{ab}$. Hence,

$$\begin{aligned} \alpha' &= \sum_{a=1}^k \sum_{b=a+1}^k h'_{ab} \\ &= \sum_{a=1}^k \sum_{b=a+1}^k ch_{ab} \\ &= \frac{\sum_{a=1}^k \sum_{b=a+1}^k h_{ab}}{\sum_{a=1}^k p_a(k-a)} \\ &= \alpha. \end{aligned}$$

Moreover, all Constraints 38, 40, 41, 42, and 43 hold since by dividing the sides of inequality we achieve the constraints in MP 28. Thus for any feasible solution for MP 28 there is a feasible

solution for LP 36 with $\alpha' = \alpha$.

$$\text{minimize } \alpha' \quad (36)$$

$$\text{subject to } \alpha' = \sum_{a=1}^k \sum_{b=a+1}^k h'_{ab} \quad (37)$$

$$p'_a \geq 0 \quad \forall 1 \leq a \leq k \quad (38)$$

$$\sum_{a=1}^k p'_a (k-a) = 1 \quad (39)$$

$$h'_{ab} \geq p'_b \quad \forall 1 \leq a < b \leq k \quad (40)$$

$$h'_{ab} \geq p'_a - 2p'_b \quad \forall 1 \leq a < b \leq k \quad (41)$$

$$h'_{ab} \geq \frac{p'_a - p'_b}{1.208} \quad \forall 1 \leq a < b \leq k \quad (42)$$

$$h'_{ab} \geq \frac{2p'_a - p'_b}{3.2} \quad \forall 1 \leq a < b \leq k \quad (43)$$

Now we write the dual program of LP 36. Finding a feasible solution for the dual program, we provide a lower bound for α . We can set the objective function of LP 36 to the minimization of $\sum_{a=1}^k \sum_{b=a+1}^k h'_{ab}$ and remove Constraint 38. Then by assigning variables $\theta, \beta_{i,j}, \gamma_{i,j}, \lambda_{i,j}, \rho_{i,j}$ to Constraints 39, 40, 41, 42, and 43 respectively, the dual LP is as follows

$$\text{maximize } \theta \quad (44)$$

$$\text{subject to } \theta(k-a) - \sum_{i=1}^{a-1} \beta_{i,a} \quad (45)$$

$$- \sum_{j=a+1}^k \gamma_{a,j} + \sum_{i=1}^{a-1} 2\gamma_{i,a} \quad (46)$$

$$- \sum_{j=a+1}^k \frac{1}{1.208} \lambda_{a,j} + \sum_{i=1}^{a-1} \frac{1}{1.208} \lambda_{i,a} \quad (47)$$

$$- \sum_{j=a+1}^k \frac{2}{3.2} \rho_{a,j} + \sum_{i=1}^{a-1} \frac{1}{3.2} \rho_{i,a} \leq 0 \quad \forall 1 \leq a \leq k \quad (48)$$

$$\beta_{a,b} + \gamma_{a,b} + \lambda_{a,b} + \rho_{a,b} \leq 1 \quad \forall 1 \leq a < b \leq k \quad (49)$$

$$\beta_{a,b}, \gamma_{a,b}, \lambda_{a,b}, \rho_{a,b} \geq 0 \quad \forall 1 \leq a < b \leq k \quad (50)$$

Table 5 provides a feasible solution for LP 44 with $k = 10$, in which $\theta = 0.612$. Thus in LP 36, $\alpha' \geq 0.61$ and therefore $\text{PoC} \geq 0.612$. \square

3.2 Cost minimization ranking duel

In this section we consider the cost minimization version of the ranking duel with linear cost function and prove a constant upper bound for the PoC_{cost} of this game, using some structural results of the minimax strategies provided in Section 3.1.

Theorem 3.14 *For any instance of the cost minimization ranking duel with linear cost function $\text{PoC}_{\text{cost}} \leq 3$.*

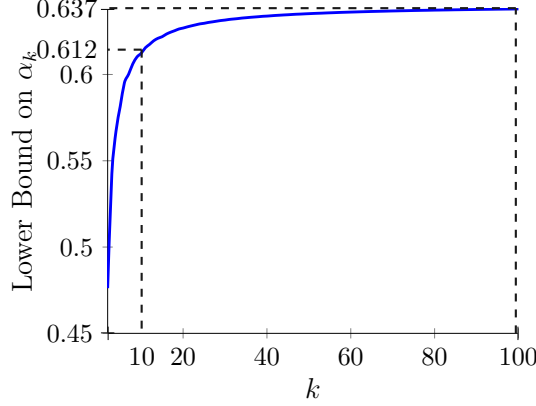


Figure 1: **Lower bound on the solution of MP 22.** While we formally prove $\alpha_{10} \geq 0.612$, this figure shows lower bounds on α_k for $2 \leq k \leq 100$ found by a computer program. Note that, by Lemma 3.11 the PoC of the ranking duel with linear valuation function is at least α_k for all $n \geq k$.

Proof. The minimum social cost is $\sum_{b=1}^n p_b b$, obtained by sorting the webpages according to their probabilities. Similar to Proposition 3.4, for every strategy \mathbf{x} the social cost is $1 + \sum_{a=1}^n \sum_{b=i+1}^n p_a \Pr_{\pi \sim \mathbf{x}}[\pi(a) < \pi(b)] + p_b \Pr_{\pi \sim \mathbf{x}}[\pi(b) < \pi(a)]$. Thus the price of competition for a minimax strategy \mathbf{x} is

$$\frac{1 + \sum_{a=1}^n \sum_{b=i+1}^n p_a \Pr_{\pi \sim \mathbf{x}}[\pi(a) < \pi(b)] + p_b \Pr_{\pi \sim \mathbf{x}}[\pi(b) < \pi(a)]}{\sum_{b=1}^n p_b b}. \quad (51)$$

For every pair of webpages a and b with $p_a \geq p_b$ we claim that in a minimax strategy \mathbf{x} , $p_a \Pr_{\pi \sim \mathbf{x}}[\pi(a) < \pi(b)] + p_b \Pr_{\pi \sim \mathbf{x}}[\pi(b) < \pi(a)] \leq 3p_b$. Note that the polytope of the minimax strategies does not differ in the welfare maximization and cost minimization model. In other words, strategy \mathbf{x} is a minimax strategy for the cost minimization ranking duel *iff* it is a minimax strategy in the welfare maximization ranking duel. Thus Lemma 3.8 also holds in the cost minimization ranking duel. By Lemma 3.8, $(\frac{p_a}{2p_b} - 1) \Pr_{\pi \sim \mathbf{x}}[\pi(a) < \pi(b)] \leq \Pr_{\pi \sim \mathbf{x}}[\pi(b) < \pi(a)]$. Simplifying the formula, we have $\frac{p_a}{2p_b} \Pr_{\pi \sim \mathbf{x}}[\pi(a) < \pi(b)] \leq \Pr_{\pi \sim \mathbf{x}}[\pi(b) < \pi(a)] + \Pr_{\pi \sim \mathbf{x}}[\pi(a) < \pi(b)]$. Since $\Pr_{\pi \sim \mathbf{x}}[\pi(b) < \pi(a)] + \Pr_{\pi \sim \mathbf{x}}[\pi(a) < \pi(b)] = 1$, we have

$$\frac{p_a}{p_b} \Pr_{\pi \sim \mathbf{x}}[\pi(a) < \pi(b)] \leq 2.$$

Adding $\Pr_{\pi \sim \mathbf{x}}[\pi(b) < \pi(a)]$ to both hand sides, and then multiplying both hand sides by p_b , we have

$$p_a \Pr_{\pi \sim \mathbf{x}}[\pi(a) < \pi(b)] + p_b \Pr_{\pi \sim \mathbf{x}}[\pi(b) < \pi(a)] \leq 2p_b + p_b \Pr_{\pi \sim \mathbf{x}}[\pi(b) < \pi(a)].$$

As $\Pr_{\pi \sim \mathbf{x}}[\pi(b) < \pi(a)] \leq 1$, we can conclude

$$p_a \Pr_{\pi \sim \mathbf{x}}[\pi(a) < \pi(b)] + p_b \Pr_{\pi \sim \mathbf{x}}[\pi(b) < \pi(a)] \leq 3p_b. \quad (52)$$

Now by writing Inequality (52) property for all pairs of webpages a and b we have

$$\sum_{a=1}^n \sum_{b=i+1}^n p_a \Pr_{\pi \sim \mathbf{x}}[\pi(a) < \pi(b)] + p_b \Pr_{\pi \sim \mathbf{x}}[\pi(b) < \pi(a)] \leq 3 \sum_{b=1}^n p_b (b-1) = 3 \left(\sum_{b=1}^n p_b b - \sum_{b=1}^n p_b \right).$$

Since $\sum_{b=1}^n p_b = 1$, by simplifying formula we have

$$\frac{3 + \sum_{a=1}^n \sum_{b=i+1}^n p_a Pr_{\pi \sim \mathbf{x}}[\pi(a) < \pi(b)] + p_b Pr_{\pi \sim \mathbf{x}}[\pi(b) < \pi(a)]}{\sum_{b=1}^n p_b b} \leq 3.$$

Thus by Equation (51), $PoC_{cost} \leq 3$. □

4 General framework

In this section we present a general framework for analyzing the price of competition in dueling games. Proving lower bounds for the price of competition in dueling games highly depends on the valuation functions and it becomes more challenging when the valuation functions are complex. However, the behavior of minimax strategies only depends on the comparison of the valuation functions rather than actual values. We leverage this fact to provide Theorem 4.1 which enables us to prove bounds for the price of competition without concerning the complexities of the valuation functions. We refer to this theorem as the *0-1 principle*.

Let (Ω, p, S, v) be a dueling game and α be a non-negative real number. We define the trigger function $v_\omega^\alpha(s)$ for a pure strategy s in the following way:

$$v_\omega^\alpha(s) = \begin{cases} 1 & \text{if } v_\omega(s) \geq \alpha \\ 0 & \text{if } v_\omega(s) < \alpha \end{cases}$$

Moreover, we define the pseudo-welfare function $SW^\alpha(s)$ as the summation of the values of the trigger functions when a player is playing strategy s with respect to α , $SW^\alpha(s) = \sum_{\omega \in \Omega} p_\omega v_\omega^\alpha(s)$. Furthermore, the pseudo-welfare function for a mixed strategy \mathbf{x} is defined as $SW^\alpha(\mathbf{x}) = E_{s \sim \mathbf{x}}[SW^\alpha(s)]$. Let PoC^α be the pseudo-welfare of the minimax strategy with the least social welfare over $SW(OPT)$ which can be formulated by $PoC^\alpha = \frac{SW^\alpha(\mathbf{x}^*)}{SW^\alpha(OPT)}$, where \mathbf{x}^* is the minimax strategy with the least social welfare and OPT is the strategy with highest social welfare. Note that optimal and minimax strategies are determined regardless of the pseudo-welfare function. For simplicity, we consider $PoC^\alpha = 1$ when $SW^\alpha(OPT) = 0$. In the following we show that the PoC of every dueling game is bounded by $\min_{\alpha \geq 0} \{PoC^\alpha\}$.

Theorem 4.1 (*0-1 principle*) *For every dueling game we have $PoC \geq \min_{\alpha \geq 0} \{PoC^\alpha\}$.*

Proof. Since $v_\omega^\alpha(s) = 1$ if and only if $v_\omega(s) \geq \alpha$, and $v_\omega^\alpha(s) = 0$ otherwise, we can formulate $v_\omega(s)$ as:

$$v_\omega(s) = \int_0^\infty v_\omega^\alpha(s) d\alpha. \tag{53}$$

Therefore, for every strategy \mathbf{x} we have:

$$\begin{aligned} SW(\mathbf{x}) &= E_{s \sim \mathbf{x}} \left[\sum_{\omega \in \Omega} p_\omega v_\omega(s) \right] \\ &= E_{s \sim \mathbf{x}} \left[\int_0^\infty \sum_{\omega \in \Omega} p_\omega v_\omega^\alpha(s) d\alpha \right] && \text{By Equation (53)} \\ &= \int_0^\infty E_{s \sim \mathbf{x}} [SW^\alpha(s)] d\alpha && \text{By the definition of } SW^\alpha(s) \end{aligned}$$

$$= \int_0^\infty \text{SW}^\alpha(\mathbf{x}) d\alpha.$$

Let $\beta = \min_{\alpha \geq 0} \{\text{PoC}^\alpha\}$. Thus, for every minimax strategy \mathbf{x}^* and $\alpha \geq 0$ we have $\beta \times \text{SW}^\alpha(\text{OPT}) \leq \text{SW}^\alpha(\mathbf{x}^*)$. Hence,

$$\begin{aligned} \beta \times \text{SW}(\text{OPT}) &= \beta \times \int_0^\infty \text{SW}^\alpha(\text{OPT}) d\alpha \\ &\leq \int_0^\infty \text{SW}^\alpha(\mathbf{x}^*) d\alpha = \text{SW}(\mathbf{x}^*), \end{aligned} \tag{54}$$

which implies $\text{PoC} \geq \beta$. \square

In the following subsections we show how we can apply the 0-1 principle to dueling games in order to present lower bounds for the PoC. In Subsection 4.1 we show that the PoC of the ranking duel is at least $\frac{1}{4}$ regardless of the valuation function. Note that, for every α , one could design a valuation function in such a way that $|v_\omega^\alpha(\mathbf{x}) - v_\omega(\mathbf{x})| \leq \epsilon$ while the optimal and minimax strategies remain the same. Therefore, we have the lowest PoC when the range of the valuation function is $[0, \epsilon] \cup [1, 1 + \epsilon]$. We use this fact to provide upper bounds for the PoC of binary search and compression duels in Subsections 4.2 and 4.3.

4.1 Ranking duel with general valuation function

Recall that in the ranking duel each position of the permutation has a valuation $f(i)$, each pure strategy of the players is a permutation of webpages $\pi = \langle \pi^{-1}(1), \pi^{-1}(2), \pi^{-1}(3), \dots, \pi^{-1}(n) \rangle$, and $\Omega = \{1, 2, \dots, n\}$ is the set of elements of uncertainty. For a webpage $\omega \in \Omega$, $v_\omega(\pi) = f(\pi(\omega))$, where $\pi(\omega)$ is the rank of ω in π . In the following, we use the 0-1 principle to show that the PoC of the ranking duel with an arbitrary valuation function is at least $\frac{1}{4}$.

Theorem 4.2 *The PoC of the ranking duel is at least $\frac{1}{4}$.*

Proof. To prove this lower bound we first show that PoC^α of this game is at least $\frac{1}{4}$ for all $\alpha \geq 0$ and then we apply the 0-1 principle and conclude that $\text{PoC} \geq \frac{1}{4}$. Let $\alpha \geq 0$ be a real number and k be the number of indices i of the permutation such that $f(i) \geq \alpha$. Therefore, for each pure strategy π , $v_{\omega_i}^\alpha = 1$ for exactly k elements ω_i and $v_{\omega_i}^\alpha = 0$ for the other $n - k$ elements. Since $p_1 \geq p_2 \geq \dots \geq p_n$,

$$\text{SW}^\alpha(\text{OPT}) \leq \sum_{i=1}^k p_i. \tag{55}$$

Also for every pure strategy π , $\text{SW}^\alpha(\pi) \leq \sum_{i=1}^k p_i$. Now let \mathbf{x}^* be the minimax strategy with the least social welfare. For each webpage i , let q_i be the probability that \mathbf{x}^* puts webpage i in a position with valuation at least α . Therefore we can formulate $\text{SW}^\alpha(\mathbf{x}^*)$ as

$$\text{SW}^\alpha(\mathbf{x}^*) = \sum_{i=1}^n p_i q_i. \tag{56}$$

Consider the mixed strategy \mathbf{x}'_i which draws a random permutation π from \mathbf{x}^* and plays permutation $\mathcal{T}(\pi)$ as follows

- If $f(\pi(i)) \geq \alpha$ then return π .
- Otherwise, Let w_1, w_2, \dots, w_k be the set of webpages such that $f(\pi(w_i)) \geq \alpha$. Choose one of the webpages in $\{w_1, w_2, \dots, w_k\}$ uniformly at random and swap the position of that webpage with the position of webpage i and return the new permutation.

Note that strategy \mathbf{x}'_i plays in such a way that $f(\pi(w_i)) \geq \alpha$, and thus $v_\omega^\alpha(\mathbf{x}'_i) = 1$. Since \mathbf{x}^* is a minimax strategy, the payoff $u^A(\mathbf{x}^*, \mathbf{x}'_i)$ should be greater than or equal to 0. This observation, follows from the fact that in an NE of the game, the payoff of both players is equal to 0, hence, every minimax strategy should guarantee a payoff of at least 0. In the following, we bound the payoff of the game between \mathbf{x}^* and \mathbf{x}'_i in terms of p_i 's and q_i 's as follows

$$u_i^A(\mathbf{x}^*, \mathbf{x}'_i) \leq -(1 - q_i)^2, \quad (57)$$

$$\forall \omega \neq i \quad u_\omega^A(\mathbf{x}^*, \mathbf{x}'_i) \leq 2q_\omega.$$

Given that the payoff of \mathbf{x}^* is at least 0, we conclude the following inequality

$$p_i(1 - q_i)^2 \leq \sum_{\omega=1}^n \frac{2p_\omega q_\omega}{k}.$$

By definition, $u_i^A(\mathbf{x}^*, \mathbf{x}'_i)$ is specified by the following formula

$$Pr_{\substack{\pi_A \sim \mathbf{x}^* \\ \pi_B \sim \mathbf{x}^*}}[v_i(\pi_A) > v_i(\mathcal{T}(\pi_B))] - Pr_{\substack{\pi_A \sim \mathbf{x}^* \\ \pi_B \sim \mathbf{x}^*}}[v_i(\pi_A) < v_i(\mathcal{T}(\pi_B))].$$

In order to prove Inequality (57), we consider the following two cases

- (i). $v_i^\alpha(\pi_A) = v_i^\alpha(\pi_B) = 0$. Since $v_i^\alpha(\mathcal{T}(\pi_B))$ is always equal to 1, we have $Pr[v_i(\pi_A) > v_i(\mathcal{T}(\pi_B))] - Pr[v_i(\pi_A) < v_i(\mathcal{T}(\pi_B))] = -1$.
- (ii). Otherwise, since π_A and π_B are both drawn from the same strategy, we know that the expected value of $Pr[v_i(\pi_A) > v_i(\pi_B)] - Pr[v_i(\pi_A) < v_i(\pi_B)]$ in this case is 0. Furthermore, $v_i(\mathcal{T}(\pi_B)) \geq v_i(\pi_B)$, thus $Pr[v_i(\pi_A) > v_i(\mathcal{T}(\pi_B))] - Pr[v_i(\pi_A) < v_i(\mathcal{T}(\pi_B))] \leq 0$.

Since for $\pi \sim \mathbf{x}^*$, $v_i^\alpha(\pi) = 1$ with probability q_i , the first case happens with probability $(1 - q_i)^2$ and the second case happens with probability $1 - (1 - q_i)^2$, we have

$$Pr_{\substack{\pi_A \sim \mathbf{x}^* \\ \pi_B \sim \mathbf{x}^*}}[v_i(\pi_A) > v_i(\mathcal{T}(\pi_B))] - Pr_{\substack{\pi_A \sim \mathbf{x}^* \\ \pi_B \sim \mathbf{x}^*}}[v_i(\pi_A) < v_i(\mathcal{T}(\pi_B))] \leq -(1 - q_i)^2,$$

which implies Inequality (57). Next, we show that $u_\omega^A(\mathbf{x}^*, \mathbf{x}'_i) \leq 2q_\omega$ for all $\omega \neq i$. Again, by definition, we have

$$u_\omega^A(\mathbf{x}^*, \mathbf{x}'_i) = Pr_{\substack{\pi_A \sim \mathbf{x}^* \\ \pi_B \sim \mathbf{x}^*}}[v_\omega(\pi_A) > v_\omega(\mathcal{T}(\pi_B))] - Pr_{\substack{\pi_A \sim \mathbf{x}^* \\ \pi_B \sim \mathbf{x}^*}}[v_\omega(\pi_A) < v_\omega(\mathcal{T}(\pi_B))].$$

Note that, $v_\omega(\mathcal{T}(\pi_B)) = v_\omega(\pi_B)$ with probability at least $1 - \frac{q_\omega}{k}$ for every $\omega \neq i$ and replacing π_B by $\mathcal{T}(\pi_B)$ will increase the payoff of the strategy \mathbf{x}^* by at most 2 (changing the payoff from -1 to +1). Therefore

$$\begin{aligned} & \left(Pr_{\substack{\pi_A \sim \mathbf{x}^* \\ \pi_B \sim \mathbf{x}^*}}[v_\omega(\pi_A) > v_\omega(\mathcal{T}(\pi_B))] - Pr_{\substack{\pi_A \sim \mathbf{x}^* \\ \pi_B \sim \mathbf{x}^*}}[v_\omega(\pi_A) < v_\omega(\mathcal{T}(\pi_B))] \right) - \\ & \left(Pr_{\substack{\pi_A \sim \mathbf{x}^* \\ \pi_B \sim \mathbf{x}^*}}[v_\omega(\pi_A) > v_\omega(\pi_B)] - Pr_{\substack{\pi_A \sim \mathbf{x}^* \\ \pi_B \sim \mathbf{x}^*}}[v_\omega(\pi_A) < v_\omega(\pi_B)] \right) \leq \frac{2q_\omega}{k}. \end{aligned}$$

Given both π_A and π_B are drawn from the same strategy,

$$\left(Pr_{\substack{\pi_A \sim \mathbf{x}^* \\ \pi_B \sim \mathbf{x}^*}}[v_\omega(\pi_A) > v_\omega(\pi_B)] - Pr_{\substack{\pi_A \sim \mathbf{x}^* \\ \pi_B \sim \mathbf{x}^*}}[v_\omega(\pi_A) < v_\omega(\pi_B)] \right) = 0, \quad (58)$$

Therefore for every $\omega \neq i$,

$$u_\omega^A(\mathbf{x}^*, \mathbf{x}'_i) = \left(Pr_{\substack{\pi_A \sim \mathbf{x}^* \\ \pi_B \sim \mathbf{x}^*}}[v_\omega(\pi_A) > v_\omega(\mathcal{T}(\pi_B))] - Pr_{\substack{\pi_A \sim \mathbf{x}^* \\ \pi_B \sim \mathbf{x}^*}}[v_\omega(\pi_A) < v_\omega(\mathcal{T}(\pi_B))] \right) \leq 2q_\omega. \quad (59)$$

By applying Inequalities (57) and (59), we have

$$\begin{aligned} u^A(\mathbf{x}^*, \mathbf{x}'_i) &= \sum_{\omega=1}^n p_\omega u_\omega^A(\mathbf{x}^*, \mathbf{x}'_i) \\ &\leq -p_i(1 - q_i)^2 + \sum_{\omega \neq i} \frac{2p_\omega q_\omega}{k} \\ &\leq -p_i(1 - q_i)^2 + \sum_{\omega=1}^n \frac{2p_\omega q_\omega}{k}. \end{aligned}$$

Since \mathbf{x}^* is a minimax strategy, $u^A(\mathbf{x}^*, \mathbf{x}'_i) \geq 0$ and thus

$$p_i(1 - q_i)^2 \leq \sum_{\omega=1}^n \frac{2p_\omega q_\omega}{k}. \quad (60)$$

By summing Inequality (60) for all $1 \leq i \leq k$ we obtain $\sum_{i=1}^k p_i(1 - q_i)^2 \leq \sum_{\omega=1}^n 2p_\omega q_\omega$. Therefore,

$$\sum_{i=1}^k p_i(1 + q_i^2 - 2q_i) \leq \sum_{\omega=1}^n 2p_\omega q_\omega$$

By moving $-2q_i$ to the right hand side of the inequality we have

$$\sum_{i=1}^k p_i(1 + q_i^2) \leq \sum_{\omega=1}^n 2p_\omega q_\omega + \sum_{i=1}^k 2p_i q_i$$

Since q_i^2 is non-negative, we can remove it from the left hand side. Thus,

$$\sum_{i=1}^k p_i \leq \sum_{\omega=1}^n 2p_\omega q_\omega + \sum_{i=1}^k 2p_i q_i \leq \sum_{i=1}^n 4p_i q_i$$

Recall that, by Inequality (55) we have $\text{SW}^\alpha(\text{OPT}) \leq \sum_{i=1}^k p_i$ and by Equation (56) we have $\text{SW}^\alpha(\mathbf{x}^*) = \sum_{i=1}^n p_i q_i$. Hence,

$$\text{PoC}^\alpha = \frac{\text{SW}^\alpha(\mathbf{x}^*)}{\text{SW}^\alpha(\text{OPT})} \geq \frac{1}{4}$$

By applying the 0-1 principle, we conclude that PoC of the ranking duel game with general valuation function is at least $\frac{1}{4}$. \square

4.2 Compression duel with general valuation function

Compression duel is a dueling games which was introduced by Immorlica et al. [19]. In this game each pure strategy of the players is a binary search tree with leaf set $\Omega = \{1, 2, \dots, n\}$. Therefore, S is the set of all binary search trees that have elements of Ω as leaves and p is a distribution of probabilities over Ω . Once a request $\omega \in \Omega$ is drawn from the probability distribution p , $v_w(s)$ of each player is determined with $f(d_s(\omega))$ where $f : \mathbb{N} \rightarrow \mathbb{R}_{\geq 0}$ is a non-increasing function and $d_s(\omega)$ is the depth of the leaf ω in s . In the following, we show that the PoC of this game can be arbitrarily close to zero.

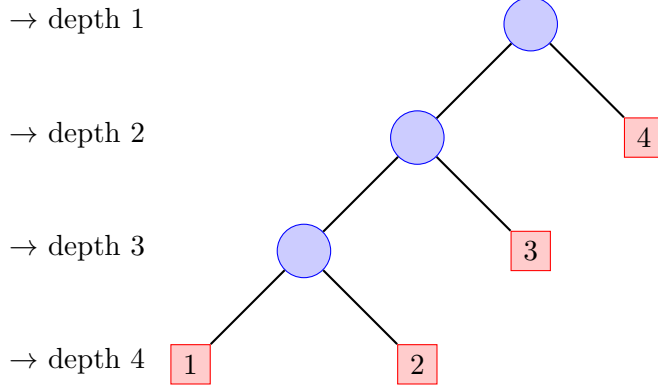


Figure 2: Depth of the webpages is 4,4,3,2, respectively.

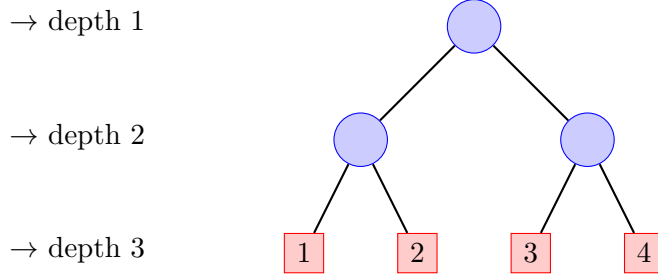


Figure 3: Depth of all the webpages is 3

Theorem 4.3 *For every $\epsilon > 0$, there exists an instance of the compression duel, with $PoC \leq \epsilon$.*

Proof. Consider a compression duel game where $\Omega = \{1, 2, 3, 4\}$ and $P = \langle \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4} \rangle$. Let the valuation function f be as follows:

$$f(d) = \begin{cases} 1 & \text{if } d \leq 2 \\ \frac{\epsilon}{16} & \text{if } d = 3 \\ 0 & \text{if } d > 3. \end{cases}$$

We show that PoC of this game is no more than ϵ . Since the binary tree in Figure 2, has one leaf with depth 2, one leaf with depth 3, and two leaves with depth 4, the social welfare of its corresponding strategy is $\frac{1}{4} \times 1 + \frac{1}{4} \times \frac{\epsilon}{16} = \frac{16+\epsilon}{64}$. Hence, we have

$$SW(OPT) \geq \frac{16 + \epsilon}{64} \tag{61}$$

Next, we prove that the pure strategy \mathbf{x}^* corresponding to the binary tree in Figure 3 is a minimax strategy of the game. To do so, we show that $u_A(\mathbf{x}^*, \mathbf{y}) \geq 0$ for every pure strategy \mathbf{y} . Since the binary tree corresponding to \mathbf{y} should have exactly 4 leaves, it has at most one leaf with depth lower than 3. Therefore one of the two condition holds for \mathbf{y} .

- (i). The binary tree corresponding to \mathbf{y} has exactly one leaf of depth 2. In this case at least two of the webpages have depth more than 3, and thus $u^A(\mathbf{x}^*, \mathbf{y}) \geq 0$.

- (ii). All the webpages have depth of 3 or more in \mathbf{y} . In this case $u_\omega^A(\mathbf{x}^*, \mathbf{y}) \geq 0$ for all $\omega = \{1, 2, 3, 4\}$. Hence $u^A(\mathbf{x}^*, \mathbf{y}) \geq 0$.

Therefore \mathbf{x}^* is a minimax strategy of the game. The social welfare of \mathbf{x}^* is equal to $(\frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4}) \times \frac{\epsilon}{16} = \frac{\epsilon}{16}$. By Inequality (61) we have:

$$\text{PoC} \leq \frac{\frac{\epsilon}{16}}{\text{SW}(\text{OPT})} \leq \frac{\frac{\epsilon}{16}}{\frac{16+\epsilon}{64}} \leq \frac{4\epsilon}{16+\epsilon} \leq \epsilon$$

□

4.3 Binary search duel with general valuation function

In this subsection we study the binary search duel and show that the PoC of this game can be $\Omega(\frac{1}{n})$. In this game $\Omega = \{1, 2, \dots, n\}$ and each pure strategy of the players is a binary tree such that its in-order traversal visits the elements of Ω in the sorted order. Moreover, $v_\omega(s)$ is determined by $f(d_s(\omega))$ where $d_s(\omega)$ denotes the depth of element ω in the binary search tree corresponding to s and $f: \mathbb{N} \rightarrow \mathbb{R}_{\geq 0}$ is a decreasing function.

Theorem 4.4 *For every $\beta > 0$ there is an instance of the binary search duel with $|\Omega| = \theta(\frac{1}{\beta})$ and $\text{PoC} \leq \beta$.*

Proof. Let $k = \lceil \lg \frac{1}{\beta} \rceil + 2$ and $\Omega = \{1, 2, \dots, n\}$ be the set of webpages where $n = 3 \times 2^k$. Moreover, let $P = \langle \frac{1}{5}, \frac{4}{5(n-1)}, \frac{4}{5(n-1)}, \dots, \frac{4}{5(n-1)} \rangle$ and the valuation function f be as follows:

$$f(d) = \begin{cases} 1 & \text{if } d = 1 \\ \epsilon & \text{if } 2 \leq d \leq k+2 \\ 0 & \text{if } d > k+2. \end{cases}$$

Where ϵ is a very small positive number. Note that, since $\lceil \lg \frac{1}{\beta} \rceil \geq 1$, n is always greater than or equal to 24. Moreover, The valuation function for the root of the tree is equal to 1. Thus, by putting the first webpage as the root of the tree we can achieve the social welfare of at least $\frac{1}{5}$. Therefore, we have:

$$\text{SW}(\text{OPT}) \geq \frac{1}{5} \tag{62}$$

Next, we show that there exists a minimax strategy of this game with social welfare equal to $\frac{4}{5(n-1)} + \frac{5(n-1)-4}{5(n-1)}\epsilon$. Consider the pure strategy \mathbf{x}^* such that plays the binary tree of Figure 4. In this strategy, depth of the first webpage is always 2 and the depth all other webpages is at most $k+2$. Therefore, the social welfare of this strategy is equal to $\frac{4}{5(n-1)} + \frac{5(n-1)-4}{5(n-1)}\epsilon$. Furthermore, we show that $u^A(\mathbf{x}^*, \mathbf{y}) \geq 0$ for all pure strategies \mathbf{y} and conclude that \mathbf{x}^* is a minimax strategy. To do so, we divide the pure strategies into two categories:

- (i). The pure strategies that have the first webpage as the root of the tree. Since the in-order traversal of the webpages in these binary trees should be from first webpage to the last webpage, all the other webpages are in the right subtree of the root. Therefore in these strategies depth of at least 2^k webpages is higher than $k+2$. Thus, $u_\omega^A(\mathbf{x}^*, \mathbf{y}) = 1$ for at least 2^k webpages and $u_1^A(\mathbf{x}^*, \mathbf{y}) = -1$. Since $n \geq 24$, we have $p_1 = \frac{1}{5} < 2^k \times \frac{4}{5(n-1)} = \frac{n}{3} \times \frac{4}{5(n-1)}$. Therefore, $u^A(\mathbf{x}^*, \mathbf{y}) \geq 0$.

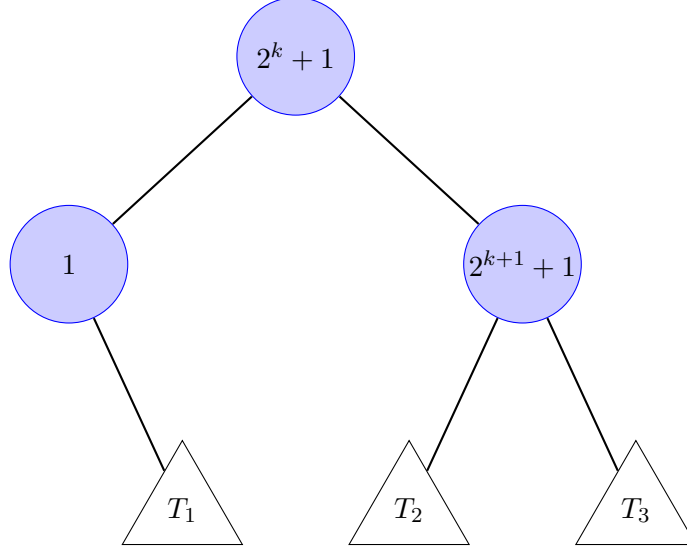


Figure 4: All the subtrees are complete binary trees with height k . Subtree T_1 contains all the webpages from 2 to 2^k , subtree T_2 contains all the webpages from $2^k + 2$ to 2^{k+1} and subtree T_3 contains all the webpages from $2^{k+1} + 2$ to 3×2^k

- (ii). The pure strategies in which depth of the first webpage is more than 1. In these strategies one webpage ω such that $p_\omega = \frac{4}{5(n-1)}$ is the root and the value of the valuation function for all other webpages is at most ϵ . Since the value of the valuation function for all the webpages in \mathbf{x}^* is at least ϵ and its root has the same probability to be requested, we have $u^A(\mathbf{x}^*, \mathbf{y}) \geq 0$.

Since $u^A(\mathbf{x}^*, \mathbf{y}) \geq 0$ for all pure strategies \mathbf{y} , \mathbf{x}^* is a minimax strategy. This implies that the following inequality holds for the PoC of this game:

$$\text{PoC} \leq \frac{\text{SW}(\mathbf{x}^*)}{\text{SW}(\text{OPT})} \leq \frac{\frac{4}{5(n-1)} + \frac{5(n-1)-4}{5(n-1)}\epsilon}{\frac{1}{5}} \leq \frac{4}{n-1} + 5\epsilon$$

Since $n \geq 24$ and ϵ is small enough, we have:

$$\text{PoC} \leq \frac{4}{n-1} + 5\epsilon \leq \frac{5}{n}$$

Recall that, $n = 3 \times 2^{\lceil \lg \frac{1}{\beta} \rceil + 2}$, therefore $\frac{5}{n} < \beta$ and thus $\text{PoC} \leq \beta$. \square

Acknowledgment

We would like to gratefully thank Rakesh Vohra and Brendan Lucier for their insightful discussions.

References

- [1] G. Aggarwal, J. Feldman, S. Muthukrishnan, and M. Pál. Sponsored search auctions with markovian users. In *WINE*, pages 621–628. 2008.

- [2] A. Ahmadinejad, S. Dehghani, M. Hajiaghayi, H. Mahini, S. Seddighin, and S. Yazdanbod. Forming external behaviors by leveraging internal opinions. In *Computer Communications (INFOCOM), 2015 IEEE Conference on*, pages 1849–1857. IEEE, 2015.
- [3] M. Ahmadinejad, S. Dehghani, M. Hajiaghayi, B. Lucier, H. Mahini, and S. Seddighin. From duels to battefields: Computing equilibria of blotto and other games. *AAAI*, 2016.
- [4] S. Albers, S. Eilts, E. Even-Dar, Y. Mansour, and L. Roditty. On nash equilibria for a network creation game. In *SODA*, pages 89–98, 2006.
- [5] N. Alon, E. D. Demaine, M. T. Hajiaghayi, and T. Leighton. Basic network creation games. *SIAM Journal on Discrete Mathematics*, 27(2):656–668, 2013.
- [6] N. Andelman, M. Feldman, and Y. Mansour. Strong price of anarchy. In *SODA*, pages 189–198, 2007.
- [7] I. Ashlagi, P. Krysta, and M. Tennenholtz. Social context games. In *Internet and Network Economics*, pages 675–683. Springer, 2008.
- [8] C. Aspremont, J. J. Gabszewicz, and J.-F. Thisse. On hotelling’s” stability in competition”. *Econometrica: Journal of the Econometric Society*, pages 1145–1150, 1979.
- [9] S. Athey and G. Ellison. Position auctions with consumer search. Technical Report 15253, National Bureau of Economic Research, 2009.
- [10] J. Bertrand. Book review of théorie mathématique de la richesse sociale and of recherches sur les principes mathématiques de la théorie des richesses. *Journal de Savants*, 67:499–508, 1883.
- [11] F. Brandt, F. Fischer, P. Harrenstein, and Y. Shoham. Ranking games. *Artificial Intelligence*, 173(2):221–239, 2009.
- [12] G. Christodoulou and E. Koutsoupias. The price of anarchy of finite congestion games. In *STOC*, pages 67–73, 2005.
- [13] T. H. Cormen, C. Stein, R. L. Rivest, and C. E. Leiserson. *Introduction to Algorithms*. McGraw-Hill Higher Education, 2nd edition, 2001.
- [14] E. D. Demaine, M. Hajiaghayi, H. Mahini, and M. Zadimoghaddam. The price of anarchy in network creation games. In *PODC*, pages 292–298, 2007.
- [15] B. Edelman, M. Ostrovsky, and M. Schwarz. Internet advertising and the generalized second-price auction: Selling billions of dollars worth of keywords. *American Economic Review*, 97(1):242–259, 2007.
- [16] A. Fabrikant, A. Luthra, E. Maneva, C. H. Papadimitriou, and S. Shenker. On a network creation game. In *PODC*, pages 347–351, 2003.
- [17] A. Ghose and S. Yang. An empirical analysis of search engine advertising: Sponsored search in electronic markets. *Management Science*, 55(10):1605–1622, 2009.
- [18] H. Hotelling. Stability in competition. *The Economic Journal*, 39(153):41–57, 1929.
- [19] N. Immorlica, A. T. Kalai, B. Lucier, A. Moitra, A. Postlewaite, and M. Tennenholtz. Dueling algorithms. In *STOC*, pages 215–224, 2011.

- [20] N. Immorlica, L. E. Li, V. S. Mirrokni, and A. S. Schulz. Coordination mechanisms for selfish scheduling. *Theoretical Computer Science*, 410(17):1589–1598, 2009.
- [21] D. Kempe and B. Lucier. User satisfaction in competitive sponsored search. In *WWW*, pages 699–710, 2014.
- [22] D. Kempe and M. Mahdian. A cascade model for externalities in sponsored search. In *Internet and Network Economics*, pages 585–596. 2008.
- [23] E. Koutsoupias and C. Papadimitriou. Worst-case equilibria. In *STACS*, pages 404–413, 1999.
- [24] D. M. Kreps. *A course in microeconomic theory*. Harvester Wheatsheaf New York, 1990.
- [25] A. Mas-Colell, M. D. Whinston, and J. R. Green. *Microeconomic theory*. Oxford university press New York, 1995.
- [26] N. Nisan, T. Roughgarden, E. Tardos, and V. V. Vazirani. *Algorithmic Game Theory*. Cambridge University Press, 2007.
- [27] T. Roughgarden. *Selfish routing and the price of anarchy*, volume 174. MIT press Cambridge, 2005.
- [28] R. Telang, U. Rajan, and T. Mukhopadhyay. The market structure for internet search engines. *Journal of Management Information Systems*, 21(2):137–160, 2004.
- [29] H. R. Varian and J. Repcheck. *Intermediate microeconomics: a modern approach*. WW Norton & Company New York, NY, 8th edition, 2010.

A A feasible solution for the dual linear program

θ	β	λ	ρ
$\theta = 0.612275$	$\beta_{1,2} = 1$	$\lambda_{1,5} = 1$	$\rho_{1,4} = 0.86974$
	$\beta_{1,3} = 1$	$\lambda_{1,6} = 1$	$\rho_{2,6} = 0.939106$
	$\beta_{1,4} = 0.13026$	$\lambda_{1,7} = 1$	$\rho_{3,7} = 0.332976$
	$\beta_{2,3} = 1$	$\lambda_{1,8} = 1$	$\rho_{3,8} = 1$
	$\beta_{2,4} = 1$	$\lambda_{1,9} = 1$	$\rho_{3,9} = 1$
	$\beta_{2,5} = 1$	$\lambda_{1,10} = 1$	$\rho_{4,7} = 0.0537254$
	$\beta_{2,6} = 0.0608937$	$\lambda_{2,7} = 1$	$\rho_{4,8} = 0.615102$
	$\beta_{3,4} = 1$	$\lambda_{2,8} = 1$	$\rho_{4,9} = 1$
	$\beta_{3,5} = 1$	$\lambda_{2,9} = 1$	$\rho_{4,10} = 0.274381$
	$\beta_{3,6} = 1$	$\lambda_{2,10} = 1$	$\rho_{5,9} = 0.422703$
	$\beta_{3,7} = 0.667024$	$\lambda_{3,10} = 1$	$\rho_{5,10} = 1$
	$\beta_{4,5} = 1$	$\lambda_{4,10} = 0.725619$	$\rho_{6,9} = 0.420799$
	$\beta_{4,6} = 1$		$\rho_{6,10} = 0.394388$
	$\beta_{4,7} = 0.946275$		
	$\beta_{4,8} = 0.384898$		
	$\beta_{5,6} = 1$		
	$\beta_{5,7} = 1$		
	$\beta_{5,8} = 1$		
	$\beta_{5,9} = 0.577297$		
	$\beta_{6,7} = 1$		
	$\beta_{6,8} = 1$		
	$\beta_{6,9} = 0.579201$		
	$\beta_{6,10} = 0.605612$		
	$\beta_{7,8} = 1$		
	$\beta_{7,9} = 1$		
	$\beta_{7,10} = 1$		
	$\beta_{8,9} = 1$		
	$\beta_{8,10} = 1$		
	$\beta_{9,10} = 1$		

Figure 5: **Feasible solution for LP 44.** In this table we present the non-zero variables of LP 44, which is the dual program of LP 36. This feasible solution gives us a lower bound for the primal linear program.